MEMORANDUM
RM-2881-PR
MARCH 1962

SOME METHODS FOR ESTABLISHING INTERPLANETARY TRANSFER ORBITS

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PREPARED FOR:
UNITED STATES AIR FORCE PROJECT RAND

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PREFACE

This Memorandum is intended to be a fundamental aid in studies of orbital mechanics. The relatively simple, straightforward computational procedures described herein can be used to establish approximate heliocentric transfer orbits and their elements. On the basis of these approximate orbits, exact interplanetary orbits can be calculated.

These computational procedures will be of use to persons engaged in space-surveillance activities and in interplanetary-mission planning.

SUMMARY

This Memorandum presents and discusses some methods for establishing heliocentric interplanetary transfer orbits. The four basic methods and their variations can be used to establish orbits having specified transfer angles, transfer times, hyperbolic excess velocities, or heliocentric departure velocities. Each method consists of a step-by-step computation procedure which utilizes the equations of two-body motion and appropriate trigonometric relations to establish the desired transfer orbit.

Each of the methods for establishing a desired transfer orbit requires an iterative process. Thus, the methods are best applied by using a large-scale digital computer. In this way numerous orbits can be established and the orbit which is optimum for some requirement can be selected. None of the methods permits a direct analytical determination of an optimum orbit.

CONTENTS

LIST OF SYMBOLS

a = semimajor axis

a_e, e_e = semimajor axis and eccentricity of Earth's orbit

a_s, e_s = semimajor axis and eccentricity of the transfer orbit

E = eccentric anomaly

E_n, S_n, P_{an} = nth positions of the Earth, vehicle, and destination planet

e = eccentricity

i_p = inclination of destination planet orbit plane to the ecliptic

i_s = inclinations of the transfer orbit to the ecliptic plane

K = arbitrary constant

k' μ = gravitational parameter

n_s = mean angular rate of the vehicle in orbit

P_p, P_e = semilatus rectum of destination planet orbit and Earth orbit

r = distance from center of force to body

r_{ed} = distance from the Sun to the Earth at departure time

r_{pa} = distance from the destination to the Sun at arrival time

r_{pp}, r_{ep} = perihelion distances of the orbits of the destination planet and Earth

t_a = arrival time

t_d = departure time

t_s = transfer time

V_{sd} = heliocentric departure velocity of the vehicle

V_{∞} = hyperbolic excess velocity

v = true anomaly

$v_s - v_{sd}$ = vehicle transfer angle

v_{sd}, v_s = true anomaly of the vehicle at departure and arrival

α = angle between the velocity vectors of Earth and the vehicle at departure time

γ_{ed} = angle between the heliocentric velocity vector of Earth and normal to the Earth-Sun line, measured in the ecliptic plane

γ_{sd} = angle between the vehicle's velocity vector and normal to the vehicle-Sun line, measured in the transfer-orbit plane

ω_p = argument of perigee of the destination planet

η = mean angular rate

Λ_{ed} = heliocentric longitude of the Earth at departure time

Λ_{pa}, λ_{pa} = heliocentric longitude and latitude of the destination planet at arrival time

Λ_{sd} = heliocentric longitude of the vehicle at departure time

I. INTRODUCTION

Interplanetary transfer orbits may be established subject to one or more of several possible constraints. For example, one may want to find all orbits which have the same characteristic departure velocity, or a fixed time of transfer may be the most important consideration.[1,2] There are several other considerations such as arrival date, true anomaly of the departure point, etc.

This memorandum presents and discusses some computation procedures for determining the elements of elliptical interplanetary transfer orbits subject to some of the above considerations. Specifically, several different approaches are presented and the computation procedure for each is given. For all the computation procedures the transfer orbit is assumed to extend between two massless points which coincide with the center of the Earth and the center of the destination planet.

The three-dimensional transfer orbits obtained using massless departure and destination planets are more realistic than those obtained using a simplified two-dimensional model of the solar system. The former are accurate enough to give a good approximation to the true values of the transfer angle, transfer time, required velocities, etc.

Four different computation procedures, with additional variants of each, to establish the heliocentric transfer orbital elements are presented and discussed. The four different computation procedures may be employed to establish transfer orbits having, respectively, fixed transfer angles, fixed transfer times, fixed characteristic departure velocities, and, finally, fixed initial heliocentric velocities. In each case only the two-body problem is

considered with the Sun as the central body. All of the methods presented require an ephemeris of planets for the determination of planetary positions. However, these positions could be computed accurately enough using the mean orbital elements of the planetary orbits.

The above requirements are only a few of the more common ones that could be considered when establishing a transfer orbit. None of the suggested computational procedures permits a direct analytical determination of a transfer which is optimum for any one of the requirements. They are intended for use in preparing a program for a large-scale digital computer which would compute numerous orbits. Then, these orbits could be studied in order to select the optimum one.

II. GEOMETRY

Because the planets rotate about the sun in mutually inclined elliptical orbits and at different angular rates, their relative orientations continuously change. The rate of change of orientation of any two planets depends on the relative size, shape, and mutual inclination of the orbits. In general, for direct transfer to an outer planet (larger semimajor axis than the departure planet) the transfer orbit will start near perihelion for transfer angles less than 360°. At the time of departure the destination planet will, in general, lead the departure planet in its motion around the Sun. For direct transfer to an inner planet the destination planet will lag and the transfer orbit will begin near aphelion. For a given transfer angle it turns out that the departure date can vary over a relatively short period of time without incurring severe penalties. In other words, the amount of the variation in the departure date depends on the allowable variation in the magnitude and direction of the velocity vector of the vehicle at both the departure and arrival dates.

In the simplified problem which treats circular, coplanar orbits, it is possible to compute heliocentric transfer orbits using the closed analytical expressions of the two-body problem. For some purposes, the orbits established in this way are satisfactory since they give a fairly good estimate of the departure and arrival conditions. However, because the planetary orbits are eccentric and inclined the orbital data obtained using the simplified model can differ greatly from that obtained using the more realistic three-dimensional model of the solar system. For example, in the two-dimensional problem the required cutoff velocity will decrease as the transfer angle approaches 180°.

In the more realistic three-dimensional model the cutoff velocity will increase rapidly as the transfer angle approaches 180° unless the arrival planet approaches the ecliptic plane at the arrival date (this will be a rare event).

III. EQUATIONS OF MOTION

All of the methods presented and discussed here utilize approximately the same set of equations which describe simple two-body motion. Only the sequence of use and emphasis of the various equations vary.

The equations listed below are in a general form. The sequence in which equations should be used and the symbols used are indicated in each method. Some of the basic equations are given here in order to keep the equations and comments in the step-by-step computation procedures to a minimum. Thus, the methods are easier to understand and easier to compare. Equations based on two-body motion which may be used to decrease the number of iterations are also given. In some of the methods, however, the use of these equations may require more computation time than is saved by reducing the number of iterations.

The pertinent equations of motion and definition of symbols are as follows:

The distance from the central body to the vehicle is

$$r = \frac{a(1 - e^2)}{1 + e \cos (v - v_o)} \qquad (1)$$

where

a = semimajor axis

e = eccentricity

v_o = initial true anomaly

v = instantaneous true anomaly

The time required to traverse a segment of an elliptical orbit is,

according to Kepler's equation

$$t - t_o = \frac{1}{n} \left[E - E_o - e (\sin E - \sin E_o)\right] \tag{2}$$

where E is the eccentric anomaly and n is the mean angular rate in the orbit. The quantities E and n are obtained from

$$E = \sin^{-1} \left(\frac{\sqrt{1 - e^2} \sin v}{1 + e \cos v}\right) \tag{3}$$

and

$$n = \frac{k' \sqrt{\mu}}{a^{3/2}} \tag{4}$$

The transfer time may be computed using Lambert's Theorem. There are four possible elliptical paths, namely, direct, aphelion, perihelion, and indirect.[3] The four equations which are used for computing the transfer time are

$$t(\text{direct}) = \frac{1}{n} \left[\eta - \eta_1 - (\sin \eta - \sin \eta_1)\right] \tag{5a}$$

$$t(\text{aphelion}) = \frac{2\pi}{n} - \frac{1}{n} \left[\eta + \eta_1 - (\sin \eta + \sin \eta_1)\right] \tag{5b}$$

$$t(\text{perihelion}) = \frac{1}{n} \left[\eta + \eta_1 - (\sin \eta + \sin \eta_1)\right] \tag{5c}$$

$$t(\text{indirect}) = \frac{2\pi}{n} - \frac{1}{n} \left[\eta - \eta_1 - (\sin \eta - \sin \eta_1)\right] \tag{5d}$$

where, in general

$$\sin \eta/2 = \frac{1}{2} \left(\frac{r_1 + r_2 + c}{a}\right)^{1/2}$$

$$\sin \eta_1/2 = \frac{1}{2} \left(\frac{r_1 + r_2 - c}{a}\right)^{1/2}$$

and

$$c = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos (v - v_o)} \tag{6}$$

For transfer angles less than or equal to 180° either Eq. (5a) or (5b) applies. For angles greater than or equal to 180° Eq. (5c) or (5d) applies.

The transfer angle can be expressed in terms of the heliocentric latitude at the arrival date and the difference in the heliocentric longitudes of departure and arrival. The equation for transfer angle is

$$v - v_o = \cos^{-1} \left[\cos \lambda \cos (\Lambda - \Lambda_o)\right] \tag{7}$$

where λ and Λ are the instantaneous heliocentric latitude and longitude of the vehicle in the heliocentric transfer orbit.

IV. OUTLINE OF METHODS OF COMPUTATION

For the planning of manned or unmanned space missions it will probably be necessary to establish the transfer orbit subject to certain constraints such as transfer time, departure date, arrival date, guidance and control, available thrust energy, etc. The constraints will depend in part on whether the mission is manned or unmanned and on the mission requirement. For example, a manned mission may require that the transfer time be limited and specified fairly accurately in advance of departure. Therefore, it will be necessary to establish a transfer orbit which will permit transfer in the specified time. Also, if the mission requires that the vehicle pass near the destination planet it may be desirable to either minimize the distance from vehicle to Earth or to minimize the velocity of the vehicle relative to the destination planet. Again, this can be done if the transfer orbit is established subject to the proper constraints.

The computation procedures are presented in four groups according to the various constraints. Within each group there are variations which are indicated by a quantity called the parameter. The groups are outlined as follows:

A. Constraint - fixed transfer angle, $(v_s - v_{sd})$, where v_{sd} and v_s are the true anomalies of the vehicle at departure and arrival

   1. Parameter - t_a
   2. Parameter - t_d
   3. Parameter - v_{sd}

B. Constraint - fixed transfer time, t_s

1. Parameter - t_d

2. Parameter - t_a

3. Parameter - V_{sd}

C. Constraint - fixed heliocentric departure velocity, V_{sd}

1. Parameter - t_d

2. Parameter - t_a

D. Constraint - fixed hyperbolic excess velocity, V_∞

1. Parameter - t_d

2. Parameter - t_a

V. PRESENTATION AND DISCUSSION OF METHODS

A.1 FIXED $(v_s - v_{sd})$, PARAMETER t_a

1. Choose t_a and obtain Λ_{pa}, λ_{pa}, and r_{pa} from an ephemeris or by computation. Λ_{pa} and λ_{pa} are the heliocentric longitude and latitude of the destination planet and r_{pa} is the distance between the Sun and destination planet at the arrival time, t_a.

2. Compute $\Lambda_{sd} = \Lambda_{pa} - \cos^{-1}\left[\cos(v_s - v_{sd})/\cos\lambda_{pa}\right]$

3. Assume $\Lambda_{ed} = \Lambda_{sd}$ and compute r_{ed} from

$$r_{ed} = \frac{P_e}{1 + e_e \cos(\Lambda_{ed} - \Lambda_{ep})}$$

where P_e and e_e are the semilatus rectum and the eccentricity of the Earth's orbit. The longitude of Earth's perihelion is Λ_{ep}.

4. Assume that $r_{sd} = r_{ed}$ and compute the minimum value of a_s, the semimajor axis of the transfer ellipse, from

$$a_s(\min) = \frac{r_{sd} + r_{pa} + c}{4}$$

where

$$c = \sqrt{r_{sd}^2 + r_{pa}^2 - 2r_{sd}r_{pa}\cos(v_s - v_{sd})}$$

5. Compute t_s from Eq. (5a) if $(v_s - v_{sd}) \leq 180^\circ$ or from Eq. (5c) if $(v_s - v_{sd}) \geq 180^\circ$.

6. Compute $t_d = t_a - t_s$ (step 5) and obtain Λ_{ed} and r_{ed}.

7. Compute $\Delta\Lambda = \Lambda_{sd} - \Lambda_{ed}$ (step 6).

If $\Delta\Lambda > 0$, decrease t_s. Use Eq. (5a) or (5c) and $a_s = a_s$ (min)+ Δa_s to compute a new value of t_s. (Note: For a_s (min), the derivative $da_s/dt_s = 0$. A $\Delta a_s = 0.1\, a_s$ (min) will yield a value for da_s/dt_s which represents approximately the change of t_s with a_s.)[1]

If $\Delta\Lambda < 0$, increase t_s. Use Eq. (5b) or (5d) and $a_s = a_s$ (min) + Δa_s to compute a new t_s.

8. Compute $t_d = t_a - t_s$ (step 7) and obtain Λ_{ed} and r_{ed}.

9. Compute $\Delta\Lambda = \Lambda_{sd} - \Lambda_{ed}$ (step 8).

If $|\Delta\Lambda| > K$ (the size of K depends on the accuracy required), change a_s (step 7) by an amount Δa_s, compute t_s, and return to step 8. The approximate change in a_s may be obtained numerically or analytically from

$$\Delta a_s = \Delta t_s / X$$

where

$$X = \sqrt{\frac{a_s}{\mu_s}}\left[\frac{3}{2}\, t_s n_s \pm (1 - \cos \eta) \tan \eta/2 \pm (1 - \cos \eta_1) \tan \eta_1/2\right] \tag{8}$$

The multiple signs in the brackets depend on the type of transfer path as follows:

Direct = -, +	Perihelion = -, -
Aphelion = +, +	Indirect = +, -

The equation for X is derived in the Appendix.

If $|\Delta\Lambda| \leq K$, the transfer orbit is established.

In A.1 the arrival point is determined by t_a and the heliocentric longitude

of the departure point is then determined by using the coordinates of the destination planet at t_a and the transfer angle $(v_s - v_{sd})$.

By assuming that $\Lambda_{ed} = \Lambda_{sd}$ and computing r_{ed}, the departure point, S_d, of the transfer orbit is fixed when we assume that $r_{sd} = r_{ed}$.

The time of transfer, t_s, is computed for different values of a_s until the longitude of the Earth at the computed time of departure, Λ_{ed}, is approximately equal to Λ_{sd}.

Figure 1 illustrates the geometry of the motion for a typical case as the iterations are performed.

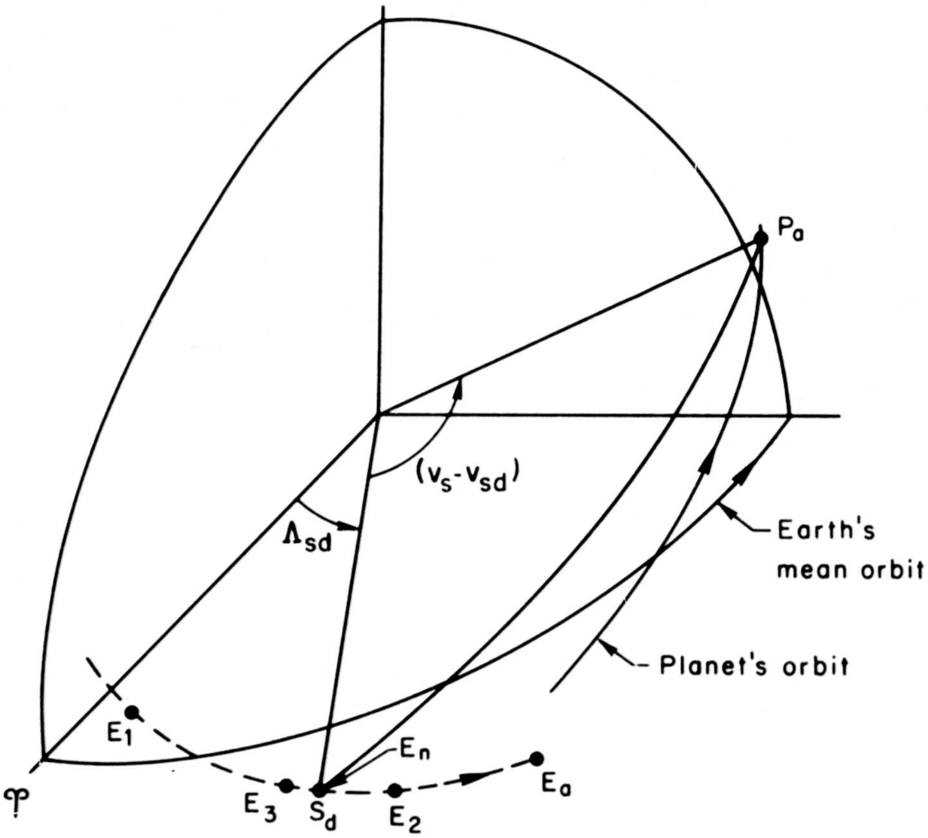

Fig. 1 — Geometry of method A.1

By starting at the time of arrival, t_a, and moving backwards in time, the motion of the vehicle and the Earth is as indicated in Fig. 1. As an example, assume that at t_a the planet is at P_a and the Earth is at E_a. During the t_s, which is computed using the initial value of a_s, i.e., a_s (min), the Earth moves from E_a to E_1 and the vehicle moves from P_a to S_d. By changing a_s so as to decrease t_s the second position of the Earth may be E_2 when the vehicle is at S_d. Additional changes in a_s will cause the nth position of the Earth, E_n, to coincide with S_d. When this is accomplished the transfer orbit is established.

A.2 FIXED $(v_s - v_{sd})$, PARAMETER t_d

1. Choose t_d and obtain Λ_{ed} and r_{ed}, the heliocentric longitude and distance of the Earth relative to the Sun at departure.
2. Compute Λ_{pa}, the heliocentric longitude of the destination planet at arrival, from

$$\Lambda_{pa} = \Omega_p + \tan^{-1} (\cos i_p \tan u_{pa})$$

where Ω_p and i_p are, respectively, the heliocentric longitude of the ascending node between the destination planet's orbit plane and the ecliptic and the inclination of the destination planet's orbit plane to the ecliptic.

The angle $u_{pa} = v_{pa} + \omega_p$ is the angle between the radius to the planet at arrival and the line of nodes, i.e., the argument of latitude, and it is obtained from

$$u_{pa} = \sin^{-1} \left[\frac{\cos (v_s - v_{sd}) \cos \beta - \cot i_p \cos (\Omega_p - \Lambda_{sd}) \sin \beta}{-\sin (\Omega_p - \Lambda_{sd}) \cos i_p \cos \beta + \cot (v_s - v_{sd}) \cos (\Omega_p - \Lambda_{sd})} \right]$$

where

$$\beta = \sin^{-1}\left[\frac{\sin i_p \sin(\Omega_p - \Lambda_{sd})}{\sin(v_s - v_{sd})}\right]$$

and

$$\Lambda_{sd} = \Lambda_{ed}$$

The equation for u_{pa} is obtained using standard spherical trigonometric formulae and therefore is not derived here.

3. Compute

$$r_{pa} = \frac{P_p}{1 + e_p \cos(u_{pa} - \omega_p)}$$

where P_p, e_p, and ω_p are known quantities and are the semilatus rectum, the eccentricity, and the argument of perihelion of the destination planet's orbit.

4. Compute the minimum value of a_s from

$$a_s\ (\min) = \frac{r_{sd} + r_{pa} + c}{4}$$

where $r_{sd} = r_{ed}$ and c is obtained as in A.1, step 4.

5. Compute t_s as in A.1, step 5.

6. Compute $t_a = t_d + t_s$ and obtain Λ_{pa}, λ_{pa}, and r_{pa}.

7. Compute

$$(v_s - v_{sd}) = \cos^{-1}\left[\cos \lambda_{pa} \cos(\Lambda_{pa} - \Omega_{sd})\right]$$

8. Compute $\Delta v_s = (v_s - v_{sd})(\text{step 7}) - (v_s - v_{sd})(\text{fixed})$.

If $\Delta v_s > 0$, decrease t_s; if $\Delta v_s < 0$, increase t_s.

The new value for t_s is computed as in A.1, step 7.

9. Compute $t_a = t_d + t_s$ (step 8) and obtain Λ_{pa}, λ_{pa}, and r_{pa}.
10. Compute $(v_s - v_{sd})$ as in step 7.
11. Compute $\Delta v = (v_s - v_{sd})(\text{step } 10) - (v_s - v_{sd})(\text{fixed})$.

If $|\Delta v_s| > K$, change a_s (step 7), compute t_s, and return to step 8. The amount that a_s should be changed, Δa_s, in order to make $|\Delta v_s| \leq K$ can be determined numerically or analytically by solving for Δa_s as follows and returning to step 8.

$$\Delta a_s = \Delta v / X$$

where

$$X = d(v_s - v_{sd})/da_s \tag{9}$$

The expression for X, which is derived in the Appendix, is as follows:

$$X = \frac{\sqrt{a_s P_p}}{r_{pa}^2}\left[\frac{3}{2} t_s n_s \pm (1 - \cos \eta) \tan \eta/2 \pm (1 - \cos \eta_1) \tan \eta_1/2\right]$$

where the multiple signs in the brackets depend on the type of transfer orbit as follows:

Direct = -, +	Perihelion = -, -
Aphelion = +, +	Indirect = +, -

In this method the departure point of the vehicle coincides with the Earth, and thus the position of the vehicle at departure is determined by choosing a t_d.

The first three steps are used to determine the vehicle arrival point on the orbit of the destination planet, S_a. In general, the point will

represent a past or future position of the destination planet depending on the time required for the vehicle to traverse the specified transfer angle.

The remaining steps are used to compute different values of $(v_s - v_{sd})_c$, using values of t_a which depend on the transfer orbit, until the computed $(v_s - v_{sd})_c$ matches the fixed $(v_s - v_{sd})_f$ -- in other words, until the position of the destination planet coincides with the position of the vehicle at arrival time.

Figure 2 shows the geometry of the motion for a typical configuration of the Earth, vehicle, and destination planet. At the departure date assume that the Earth and vehicle are at position E_d, S_d and the destination planet is at P_d. A point on the destination planet's orbit, S_a, is found which makes the angle between the radius to the point, r_{pa}, and the radius to the Earth and vehicle equal to the specified $(v_s - v_{sd})_f$. Next, an a_s

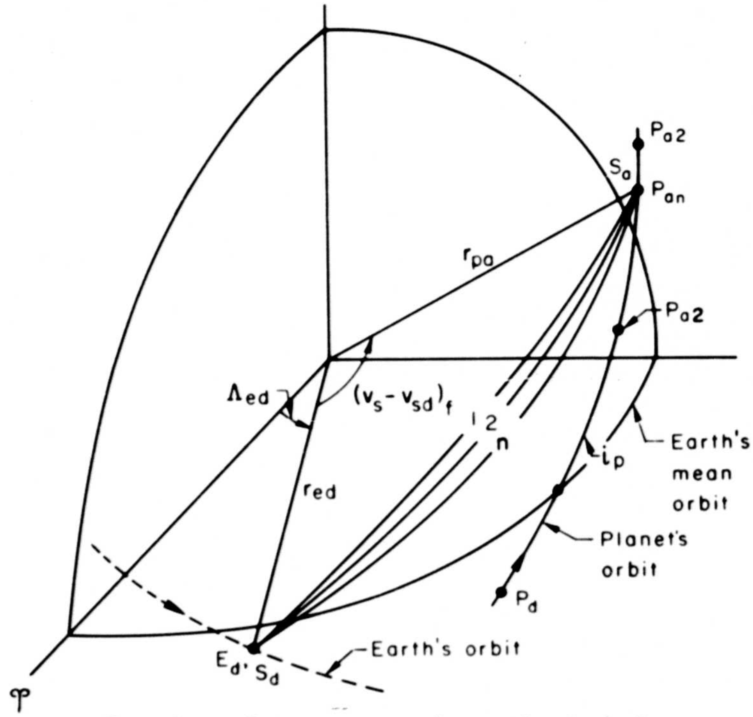

Fig. 2 — Geometry of method A.2

is chosen, in this case the minimum a_s, and the t_s is computed for transfer path 1. During the t_s the planet moves from position P_d to position P_{a1}. From Fig. 2 it is clear that the $(v_s - v_{sd})_c$ between a radius to P_{a1} and S_d does not equal $(v_s - v_{sd})_f$. Thus, another t_s is computed using a modified a_s, which results in transfer path number 2. When the vehicle reaches S_a using this path the planet may be at P_{a2} and the difference between the specified and computed $(v_s - v_{sd})$ has decreased. The process continues until the nth position of the planet, P_{an}, and S_a are approximately coincident, i.e., when the specified and computed $(v_s - v_{sd})$ are approximately equal.

A.3 FIXED $(v_s - v_{sd})$, PARAMETER v_{sd}

1. Choose v_{sd}.
2. Choose t_a and obtain Λ_{pa}, λ_{pa}, and r_{pa}.
3. Compute $\Lambda_{sd} = \Lambda_{pa} - \cos^{-1}\left[\cos (v_s - v_{sd})/\cos \lambda_{pa}\right]$, where Λ_{sd} is the heliocentric longitude of the velocity at departure.
4. Assume that $\Lambda_{ed} = \Lambda_{sd}$ and compute r_{ed} from Eq. (1).
5. Compute a_s and e_s, the eccentricity of the transfer ellipse, using the equations for r_{pa} and r_{ed}, i.e., Eq. (1).
6. Compute t_s using Eq. (2).
7. Compute $t_d = t_a - t_s$ and obtain Λ_{ed} and r_{ed}.
8. Compute $\Delta\Lambda = \Lambda_{sd} - \Lambda_{ed}$.

   If $|\Delta\Lambda| > K$, change t_a by an amount Δt_a where

$$\Delta t_a = \frac{\Delta(\Delta\Lambda)}{\sqrt{\mu_s}\left(\frac{\sqrt{P_e}}{r_{ed}^2} - \frac{\sqrt{P_p}}{r_{pa}^2}\right)} \tag{10}$$

and return to step 2.

If $\Delta\Lambda \leq K$, the transfer orbit is established.

In the above procedure, the time of arrival, t_a, is varied until the departure longitudes of the vehicle and the Earth are approximately equal.

By assuming that $\Lambda_{ed} = \Lambda_{sd}$ and that $r_{sd} = r_{ed}$, the position of the vehicle at each departure date, which is the start of the heliocentric transfer orbit, will be on the Earth's orbit. Since $(v_s - v_{sd})$ is constant, a change in the t_a will change not only the arrival position but also the departure position of the vehicle. Since the angular rate of the Earth around the Sun differs from that of vehicle departure position, a change in t_a will result in a change in the difference of the longitudes of the Earth and vehicle at departure.

Figure 3 shows how the positions of the Earth and vehicle at departure approach coincidence as the number of iterations increases. The angle between the radii from the Sun to S_n and P_{an} is a fixed value, namely $(v_s - v_{sd})$. The letters E, S, and P denote positions of the Earth, vehicle, and the destination planet, respectively.

Initially, the required position of the destination planet at arrival is P_{a1}. The vehicle and Earth are found to be at positions S_{d1} and E_{d1} at departure.

The second departure position of the vehicle, S_{d2}, is computed using the second arrival position of the destination planet, P_{a2}. The corresponding

position of the Earth at departure is E_{d2}.

The time of arrival is varied until the difference in the departure longitudes of the Earth and vehicle is approximately zero. Assume that this condition is met when the third value of t_a is computed. Then, the vehicle and Earth will be at point E_{d3}, S_{d3} at departure, the vehicle will intercept the destination planet at position P_{a3}, and the transfer orbit is established.

The usefulness of v_{sd} as a parameter is not obvious as in the case of t_d or t_a; however, it is a useful tool in that a transfer orbit can be established with a relatively short computation procedure. Also, the elevation angle of the vehicle's velocity vector at departure, γ_{sd}, changes directly with v_{sd}. In general, as v_{sd} increases, γ_{sd} will increase, and as a consequence, required hyperbolic excess or cutoff velocity will increase.

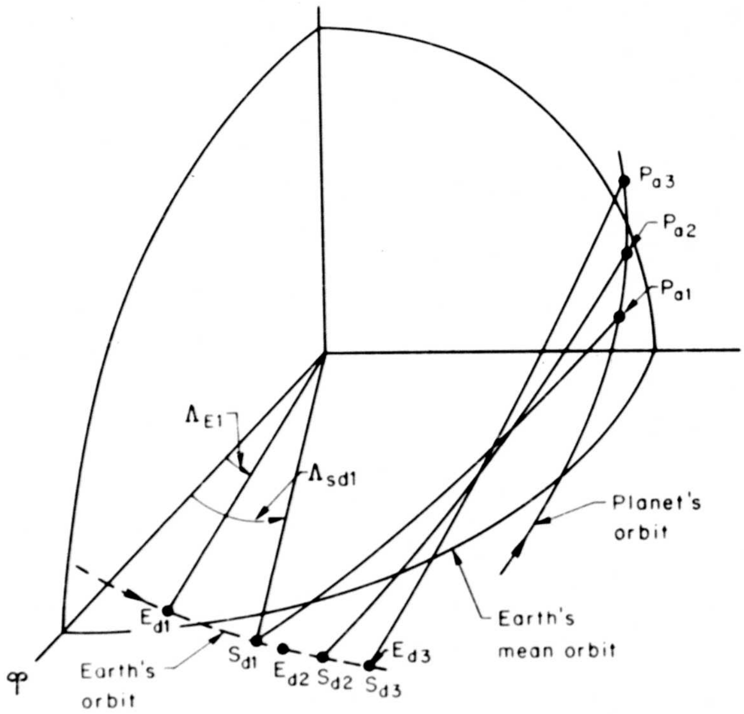

Fig. 3 — Geometry of method A.3

B.1 FIXED t_s, PARAMETER t_d

1. Choose t_d and obtain Λ_{ed} and r_{ed}.

2. Compute $t_a = t_d + t_s$ and obtain Λ_{pa}, λ_{pa}, and r_{pa}.

3. Compute $(v_s - v_{sd})$ from

$$(v_s - v_{sd}) = \cos^{-1}\left[\cos \lambda_{pa} \cos (\Lambda_{pa} - \Lambda_{sd})\right]$$

where

$$\Lambda_{sd} = \Lambda_{ed}$$

4. Compute the minimum value of a_s from

$$a_s \text{ (min)} = \frac{r_{sd} + r_{pa} + c}{4}$$

where $r_{sd} = r_{ed}$ and c is obtained from Eq. (6).

5. Compute t_s as in A.1, step 5.

6. Compute $\Delta t_s = t_s$ (step 5) - t_s (fixed).

   If $\Delta t_s \neq 0$, compute a new t_s as in A.1, step 7.

7. Compute $\Delta t_s = t_s$ (step 6) - t_s (fixed).

   If $|\Delta t_s| > K$, change a_s (step 6) by an amount Δa_s, compute t_s and Δt_s.

   Repeat step 7 until the orbit is established, i.e., until $|\Delta t_s| \leq K$.

   The amount of change for a_s may be computed as in A.1, step 9.

With t_s a given constant, a choice of t_d determines the required t_a. The heliocentric coordinates of the Earth at t_d and those of the destination planet at t_a represent two fixed points separated by a constant $(v_s - v_{sd})$ which is determined by the coordinates.

In order for the vehicle to intercept the destination planet at t_a, the actual t_s, which depends on the transfer orbit used, must equal the given t_s. This is accomplished by changing a_s, which causes a change in the actual t_s.

In Fig. 4 the point E_d, S_d represents the position of the Earth and vehicle at t_d. The point S_a represents the position of the destination planet at the required t_a which is fixed by the given t_s and chosen t_d.

Assume that the semimajor axis of transfer orbit number 1 is a(min) and that the corresponding t_s is larger than the given t_s. Then, it is clear that the vehicle will arrive at position S_a too late and the destination planet will have moved to position P_{a1}. If the correct equation for t_s is used with an increased a_s, the actual t_s will decrease and the

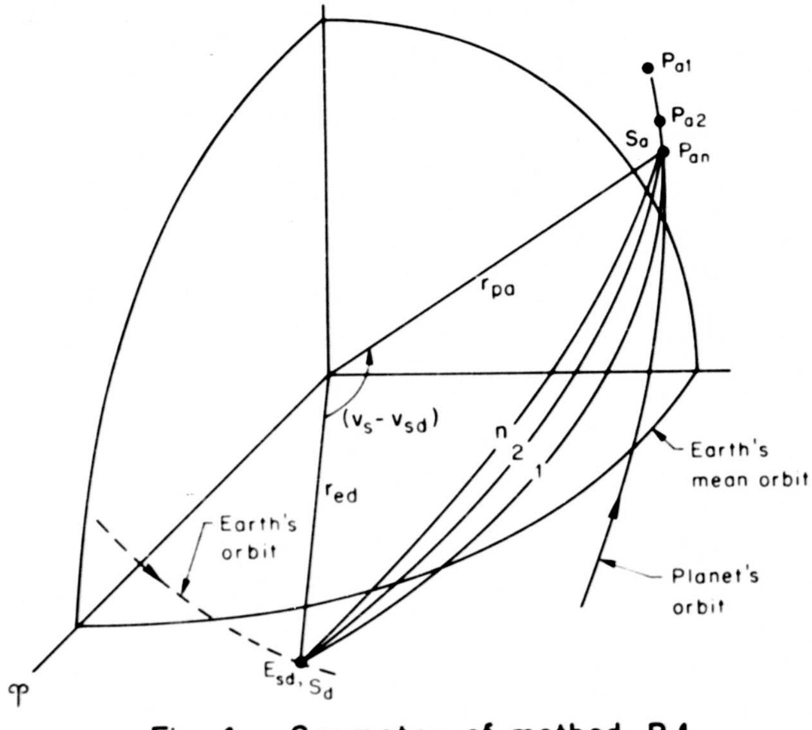

Fig. 4 — Geometry of method B.1

vehicle may arrive on transfer orbit number 2 either too early or too late. According to Fig. 4, arrival is too late, i.e., t_s is too large, and the destination planet is at P_{a2} when the vehicle is at S_a. Further changes in a_s will result in a t_s which will permit the vehicle to travel on transfer orbit number n and intercept the destination planet at S_a.

B.2 FIXED t_s, PARAMETER t_a

1. Choose the t_a and obtain Λ_{pa}, λ_{pa}, and r_{pa}.

2. Compute $t_d = t_a - t_s$ and obtain Λ_{ed} and r_{ed}.

3-7. These steps are the same as in B.1.

Since this method differs only slightly from B.1 where t_d was the parameter, the discussion of the method is similar to that of B.1, and Fig. 4 serves to illustrate the geometry.

A choice of the t_a fixes the rendezvous position of the vehicle and destination planet at point S_a. Since the t_s is fixed, a choice of t_a determines uniquely the required t_d. Thus, the required departure position of the vehicle, which must coincide with the Earth's position at departure, is determined.

As in B.1, a_s is varied until the computed t_s matches the given t_s. When this occurs, the Earth and vehicle are at the point E_d, S_d at the required departure date and the transfer orbit is established.

B.3 FIXED t_s, PARAMETER v_{sd}

1. Choose v_{sd}.

2. Choose t_d and compute

$$t_a = t_d + t_s$$

3. Obtain Λ_{ed}, r_{ed}, and Λ_{pa}, λ_{pa}, r_{pa} for t_d and t_a, respectively.

4. Compute v_s from

$$v_s = v_{sd} + \cos^{-1}\left[\cos \lambda_{pa} \cos (\Lambda_{pa} - \Lambda_{sd})\right]$$

where

$$\Lambda_{sd} = \Lambda_{ed}$$

5. Compute e_s and a_s using Eq. (1).

6. Compute the minimum value of a_s from

$$a_s(\min) = \frac{1}{4} (r_{sd} + r_{pa} + c)$$

where $r_{sd} = r_{ed}$ and c is obtained using Eq. (6).

7. Compute t_s for $a_s(\min)$ using Eq. (5a) or (5c) depending on the size of $(v_s - v_{sd})$.

8. Compute $\Delta t_s = t_s$ (step 7) - t_s (fixed).

If $\Delta t_s \neq 0$ use either Eq. (5a) or (5b), depending on the type of transfer orbit desired, with a_s (step 5) to compute t_s if $(v_s - v_{sd}) \leq 180^\circ$. Similarly, if $(v_s - v_{sd}) > 180^\circ$, use either Eq. (5c) or (5d).

If $\Delta t_s > 0$, decrease t_s (step 7) by using a_s (step 5) with Eq. (5a) or (5c) depending on the size of $(v_s - v_{sd})$.
If $\Delta t_s < 0$, increase t_s (step 7) by using a_s (step 5) with Eq. (5b) or (5d) depending on the size of $(v_s - v_{sd})$.

9. Compute $\Delta t_s = t_s$ (step 8) - t_s (fixed).
If $|\Delta t_s| > K$, change t_d by an amount Δt_d and return to step 2.

$$\Delta t_d = \Delta t_s / X$$

where

$$X = \frac{\sqrt{a_s p_p}}{r_{pa}} \left\{ \left(\frac{e_s \left[a_s (1 + e_s^2) - r_{sd} \right] \sin v_s}{(1 - e_s^2)(r_{sd} - r_{pa})} \right) \left[\frac{3}{2} n_s^2 \pm (1 - \cos \eta) \tan \frac{\eta}{2} \right.\right.$$

$$\left.\left. \pm (1 - \cos \eta_1) \tan \frac{\eta_1}{2} \right] + \frac{r_{sd}}{2c} \sin (v_s - v_{sd}) \left[\pm \tan \frac{\eta}{2} \pm \tan \frac{\eta_1}{2} \right] \right\} \quad (11)$$

The multiple signs in the brackets depend on the type of transfer path as follows:

Direct = -, + and +, + Perihelion = -, - and +, -

Aphelion = +, + and -, + Indirect = +, - and -, -

The equation for X is derived in the Appendix.

The discussion concerning the usefulness of v_{sd} as a parameter is given in method A.3.

The geometry of this method is illustrated by Fig. 5.

A selection of t_d gives a position for the Earth at departure E_{d1} and because t_s is fixed, t_a is determined, and consequently the coordinates of the planet position, P_{a1}, at the arrival time are known.

The time required for the vehicle to travel from E_{d1} to P_{a1} is computed using a prescribed v_{sd}. If the computed t_s differs from the fixed t_s, the vehicle will not intercept the planet at P_{a1}. Since the t_s will, in this case, vary directly with $(v_s - v_{sd})$, a change in $(v_s - v_{sd})$ will cause a change in t_s.

A second value of t_d, and consequently t_a, will fix the Earth at point E_{d2} and the planet at P_{a2}. Since the angular rates of the planets around the Sun are different, the angle between the radii to points E_{d2} and P_{a2} will not equal the angle between the radii to points E_{d1} and P_{a1}. Consequently, the t_s will be different.

By varying t_d the correct relative orientation of the departure and destination planets can be found which will give the correct t_s.

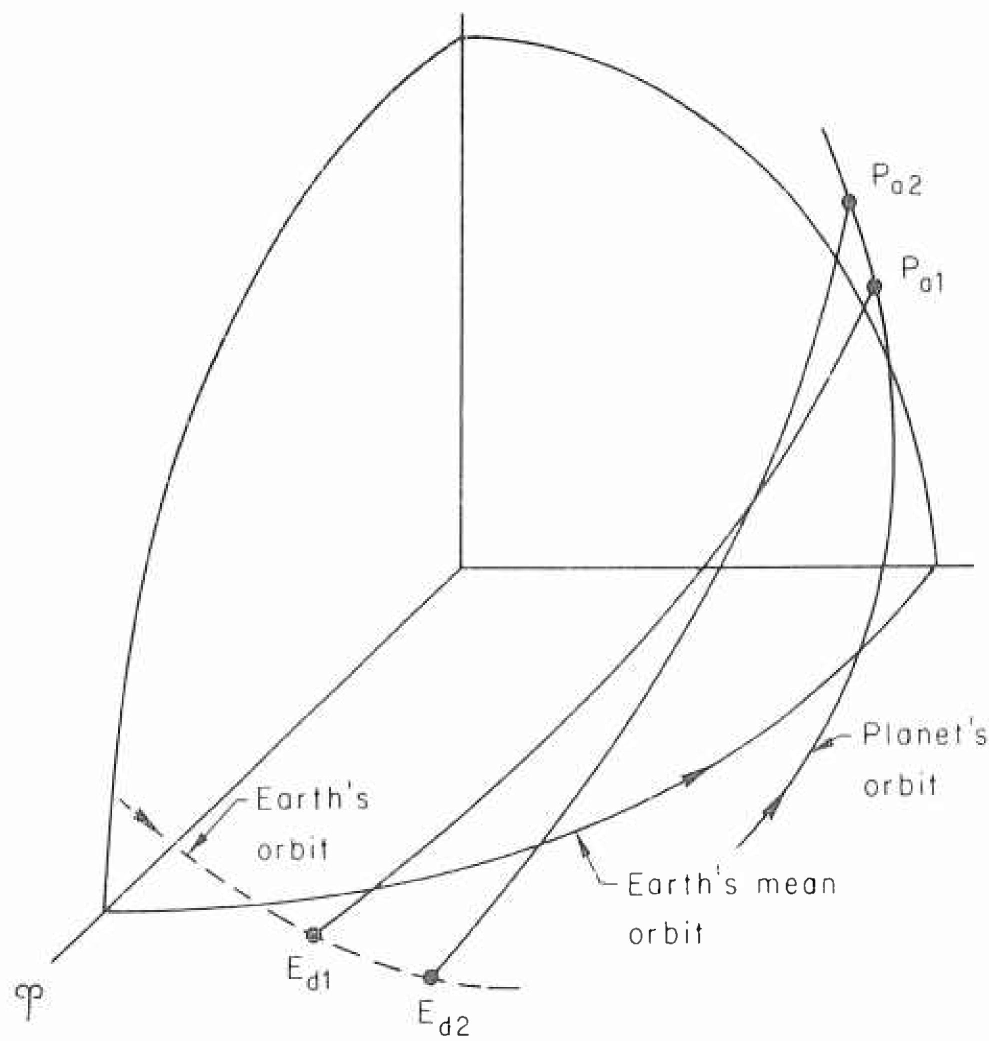

Fig. 5 — Geometry of method B.3

C.1 FIXED V_{sd}, PARAMETER t_d

1. Choose t_d and obtain Λ_{ed} and r_{ed}.

2. Compute a_s from

$$a_s = \frac{\mu_s r_{sd}}{2\mu_s - r_{sd} V_{sd}^2}$$

where $r_{sd} = r_{ed}$.

3. Choose t_a and obtain Λ_{pa}, λ_{pa}, and r_{pa}.

4. Compute $t_s = t_a - t_d$.

5. Compute $(v_s - v_{sd})$ from

$$(v_s - v_{sd}) = \cos^{-1}\left[\cos \lambda_{pa} \cos (\Lambda_{pa} - \Lambda_{sd})\right]$$

6. Compute the minimum value of a_s from

$$a_s(\text{min}) = 1/4\ (r_{sd} + r_{pa} + c)$$

where c is obtained from Eq. (6).

7. Compute $\Delta a_s = a_s(\text{min}) - a_s$ (step 2).

If $\Delta a_s > 0$, change t_a by Δt_a where

$$\Delta t_a = \Delta a_s / X$$

where

$$X = \frac{\sqrt{\mu_s P_p}\ r_{sd}}{4c\ r_{pa}} \sin (\Lambda_{pa} - \Lambda_{sd}) \tag{12}$$

and return to step 3. The equation for X is dervied in the Appendix.

If $\Delta a_s \leq 0$, proceed to step 8.

8. Compute t_s for a_s(min) using Eq. (5a) or (5c) depending on the size of $(v_s - v_{sd})$.

9. Compute $\Delta t_s = t_s$ (step 8) - t_s (step 4).

   If $\Delta t_s = 0$, either Eq. (5a) or (5b), depending on the type of transfer orbit desired, can be used with a_s (step 2) to compute t_s if $(v_s - v_{sd}) \leq 180^\circ$. Similarly, if $(v_s - v_{sd}) \geq 180^\circ$, either Eq. (5c) or (5d) would be used.

   If $\Delta t_s > 0$, decrease t_s (step 8) by using a_s (step 2) with Eq. (5a) or (5c) depending on the size of $(v_s - v_{sd})$.

   If $\Delta t_s < 0$, increase t_s (step 8) by using a_s (step 2) with Eq. (5b) or (5d) depending on the size of $(v_s - v_{sd})$.

10. Compute $\Delta t_s = t_s$ (step 9) - t_s (step 4).

    If $|\Delta t_s| > K$, change t_a by Δt_a and return to step 3.

$$\Delta t_a = \Delta t_s / X$$

where

$$X = \frac{\sqrt{a_s P_p} r_{sd}}{2c\ r_{pa}} \sin\left(\Lambda_{pa} - \Lambda_{sd}\right)\left(\pm \tan \eta/2 \pm \tan \eta_1/2\right) - 1 \qquad (13)$$

The multiple signs in the parentheses depend on the type of transfer path as follows:

Direct = +, +	Perihelion = +, -
Aphelion = -, +	Indirect = -, -

The equation for X is derived in the Appendix.

If $|\Delta t_s| \leq K$ the transfer orbit is established.

As in A.1, step 9, the equation for X will depend on the size of $(v_s - v_{sd})$ and the type of transfer path.

This computation procedure may be used to obtain transfer orbits for given heliocentric departure velocities for a specified departure date.

By using t_d as a parameter and assuming that the Earth and vehicle are coincident at departure, a search for the correct trajectory is made by varying t_a until the vehicle intercepts the destination planet. For V_{sd} and t_d some arrival times cannot be realized, i.e., the a_s which is determined by V_{sd} and t_d will be less than the required minimum which takes into account the distance to the destination planet at arrival and the transfer angle. Thus, a change in t_a will be required.

Figure 6 shows the geometry of motion for a typical case which utilizes this computation procedure.

The departure position of the Earth and vehicle, E_d, S_d, is determined by a choice of t_d. The time of arrival is arbitrarily chosen and the heliocentric coordinates of the destination planet are obtained. Assume that the planet is at point S_a at the time of arrival. The coordinates of points S_d and S_a are used to compute the minimum value of a_s. This permits one to determine if the given a_s is large enough to yield a transfer orbit. If the given a_s is less than the minimum a_s (see step 7) a new value of t_a is determined. Otherwise a t_s is computed, using the minimum a_s, which corresponds to transfer orbit number one. If the t_s is too large the planet may be at P_{a1} when the vehicle arrives at S_a. The relative position of P_{a1} and S_a permits one to select the correct equation (see Eq. 5) for t_s when transfer orbit number two is determined using the given a_s.

In general the first value of t_a will not be correct and several values of t_a will be used before the correct transfer orbit is established.

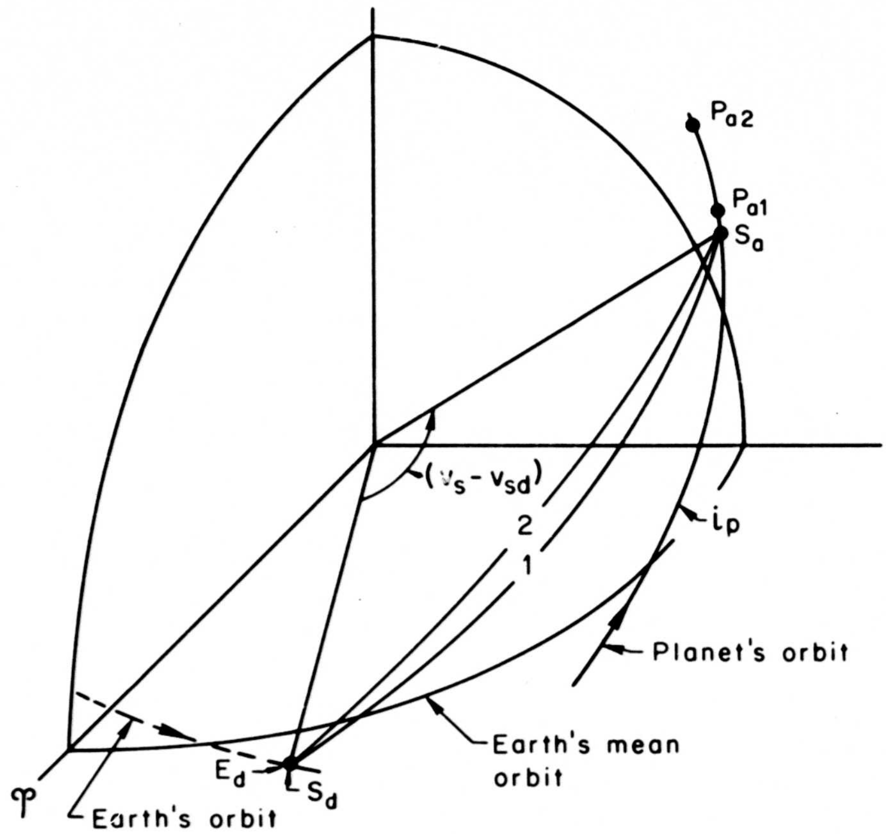

Fig. 6 — Geometry of method C.1

C.2 FIXED V_{sd}, PARAMETER t_a

1. Choose t_a and obtain Λ_{pa}, λ_{pa}, and r_{pa}.

2. Choose t_d and obtain Λ_{ed} and r_{ed}.

3. Compute $t_s = t_a - t_d$.

4. Compute a_s from

$$a_s = \frac{\mu \, r_{sd}}{2\mu - r_{sd}V_{sd}^2}$$

where $r_{sd} = r_{ed}$.

5. Compute $(v_s - v_{sd})$ from

$$(v_s - v_{sd}) = \cos^{-1}\left[\cos \lambda_{pa} \cos (\Lambda_{pa} - \Lambda_{sd})\right]$$

6. Compute the minimum value of a_s from

$$a_s(\min) = \frac{1}{4}(r_{sd} + r_{pa} + c)$$

where c is obtained using Eq. (6).

7. Compute $\Delta a_s = a_s(\min) - a_s(\text{step } 4)$.

If $\Delta a_s > 0$, change t_d by an amount Δt_d and return to step 2.

$$\Delta t_d = \Delta a_s / X$$

where

$$X = \frac{\sqrt{\mu_s P_e}\, r_{pa}}{4c\, r_{sd}} \sin (\Lambda_{pa} - \Lambda_{sd}) \tag{14}$$

The derivation of the equation for X is similar to the derivation of

the equation for X in C.1, step 7 and therefore is not included in the Appendix.

If $\Delta a_s \leq 0$, proceed to step 8.

8. Compute t_s for $a_s(\min)$ using Eq. (5a) or (5c) depending on the size of $(v_s - v_{sd})$.

9. Compute $\Delta t_s = t_s$ (step 8) - t_s (step 3).

   If $\Delta t_s = 0$ either Eq. (5a) or (5b), depending on the type of transfer orbit desired, can be used with a_s (step 4) to compute t_s if $(v_s - v_{sd}) \leq 180^\circ$. Similarly, if $(v_s - v_{sd}) > 180^\circ$, either Eq. (5c) or (5d) can be used.

   If $\Delta t_s > 0$, decrease t_s (step 8) by using a_s (step 4) with Eq. (5a) or (5c) depending on the size of $(v_s - v_{sd})$.

   If $\Delta t_s < 0$, increase t_s (step 8) by using a_s (step 4) with Eq. (5b) or (5d) depending on the size of $(v_s - v_{sd})$.

10. Compute $\Delta t_s = t_s$ (step 9) - t_s (step 3).

   If $|\Delta t_s| > K$, change t_d by Δt_d and return to step 2.

$$\Delta t_d = \Delta t_s / X$$

where

$$X = -\frac{\sqrt{a_s P_e}\; r_{pa}}{2 r_{sd} c} \sin(\Lambda_{pa} - \Lambda_{sd}) \left[\pm \tan\frac{\eta}{2} \pm \tan\frac{\eta_1}{2}\right] + 1 \qquad (15)$$

The signs of the bracketed terms are determined as in C.1, step 10. The derivation of the equation for X is similar to that for Δt_a in C.1, step 10, and therefore is not given.

If $|\Delta t_s| \leq K$, the transfer orbit is established.

This method of establishing a transfer orbit is similar to C.1. The major difference is that the time of arrival is fixed and the departure date is used as a variable. Because t_d is varied it is necessary to compute a new value of a_s for each iteration, while in C.1 a choice of t_d fixed the value of a_s for all the computations.

The time of arrival is chosen and we assume that the vehicle is coincident with the destination planets at arrival. Next t_d is varied until a transfer orbit is found which will make the departure coordinates of the Earth and vehicle equal.

Similar to C.1, some departure dates cannot be used for the given V_{sd} and selected t_a. That is, the required V_{sd} or a_s may exceed the given values. In this event it is necessary to select another departure date so as to reduce the required a_s.

Figure 7 shows the geometry and relative positions of the planets and vehicle using this computation procedure.

The geometry of the motion is easy to understand if we start at t_a and move the planets and vehicle backwards in time. At t_a the destination planet and vehicle are at point S_a, P_a on the destination planet's orbit and the Earth is at point E_a. Assume that transfer orbit 1 corresponds to the minimum value of a_s. During the corresponding time t_s the vehicle moves to point S_d while the Earth may move to point E_{d1}. In that case the t_s is too large and consequently the computed t_d is too small (early).

The relative positions of the Earth and vehicle at departure, i.e., E_{d1} and S_d, indicate which part of Eq. (5) should be used with the given a_s to compute a new value of t_s. If orbit 2 corresponds to the given a_s, then the Earth may be at E_{d2} when the vehicle is at S_d. Since E_{d2} and S_d are

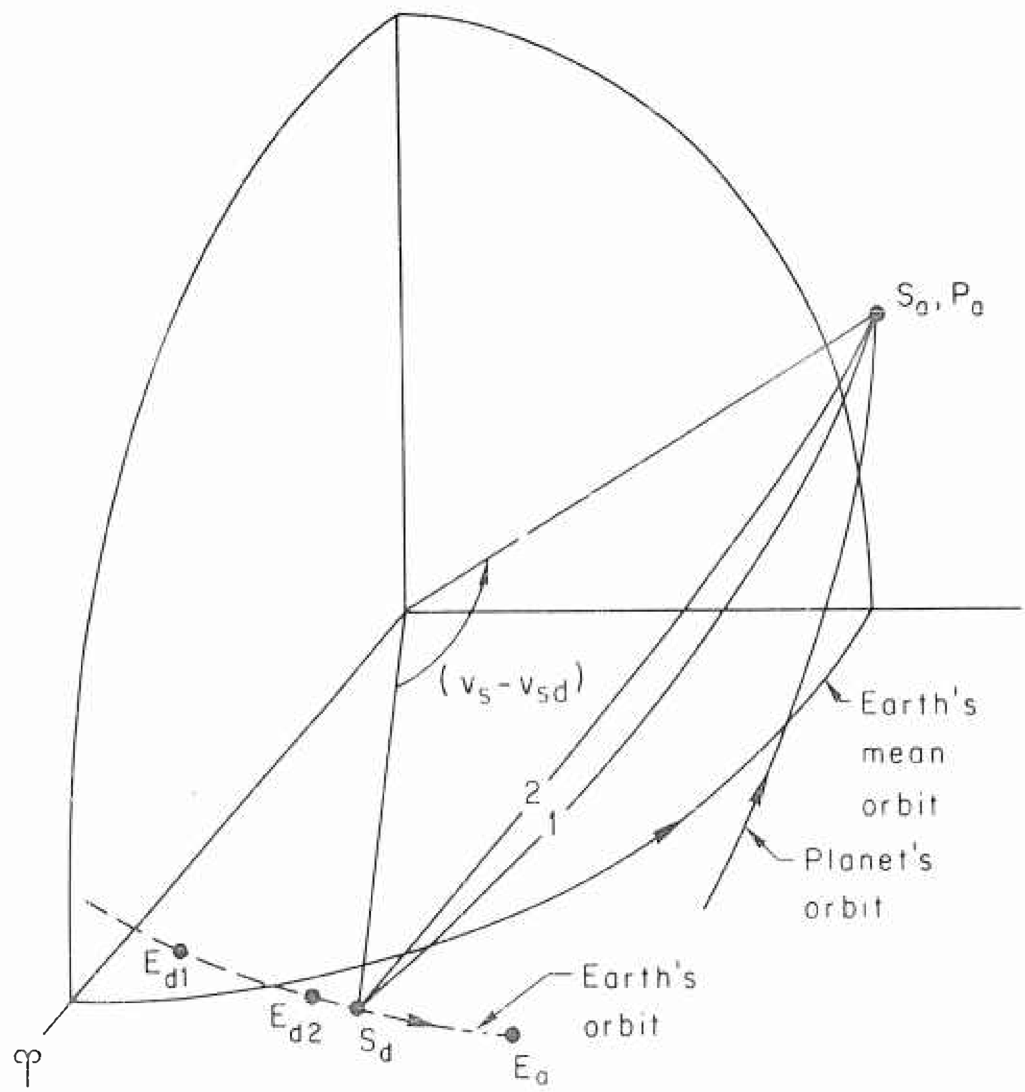

Fig. 7 — Geometry of method C.2

not coincident the selected t_d must be changed, i.e., point S_d must be at some other point on the Earth's orbit. The new value of t_d may be determined by iteration or by using the approximate equation for Δt_d, step 7.

The t_d is varied until the positions of the Earth and vehicle coincide at t_d.

D.1 FIXED V_∞, PARAMETER t_d

1. Choose t_d and obtain Λ_{ed} and r_{ed}.

2. Choose t_s.

3. Compute $t_a = t_d + t_s$ and obtain Λ_{pa}, λ_{pa}, and r_{pa}.

4. Compute $(v_s - v_{sd}) = \cos^{-1}\left[\cos \lambda_{pa} \cos (\Lambda_{pa} - \Lambda_{sd})\right]$

   where

$$\Lambda_{sd} = \Lambda_{ed}$$

5. Choose v_{sd} and compute v_s from

$$v_s = (v_s - v_{sd})(\text{step } 5) + v_{sd}$$

6. Compute e_s and a_s using Eq. (1).

7. Compute the minimum a_s from

$$a_s(\min) = \frac{1}{4}\,(r_{sd} + r_{pa} + c)$$

   where $r_{sd} = r_{ed}$ and c is obtained from Eq. (6).

8. Compute t_s using $a_s(\min)$ and Eq. (5a) or (5c) depending on the size of $(v_s - v_{sd})$.

9. Compute $\Delta t_s = t_s$ (step 8) - t_s (step 2) and compute t_s as in method C.2, step 9.

10. Compute $\Delta t_s = t_s$ (step 9) - t_s (step 2).

    If $|\Delta t_s| > K$, change v_{sd} (step 5) by Δv_{sd} and return to step 5.

$$\Delta v_{sd} = \Delta t_s / X$$

    where

$$X = \sqrt{\frac{a_s}{\mu_s}} \frac{e_s}{1 - e_s^2} \left[\frac{a_s(1 + e_s^2) - r_{sd}}{r_{sd} - r_{pa}} (r_{pa} \sin v_s - r_{sd} \sin v_{sd}) \right.$$

$$\left. - r_{sd} \sin v_{sd} \right] x \left[+ \frac{3}{2} n_s t_s \pm (1 - \cos \eta) \tan \eta/2 \right.$$

$$\left. \pm (1 - \cos \eta_1) \tan \eta_1/2 \right] \tag{16}$$

The multiple signs in the brackets depend on the type of transfer path as follows:

Direct = -, +	Perihelion = -, -
Aphelion = +, +	Indirect = +, -

The equation for X is derived in the Appendix.

If $|\Delta t_s| \leq K$, proceed to step 11.

11. Compute the Earth's velocity vector at departure from

$$V_{ed} = \sqrt{\mu_s \left(\frac{2}{r_{ed}} - \frac{1}{a_e}\right)}$$

where a_e is the semimajor axis of the Earth's orbit.

12. Compute the elevation angle of V_{ed} from

$$\gamma_{ed} = \cos^{-1} \left[\pm \sqrt{\frac{a_e^2 (1 - e_e^2)}{r_{ed} (2a_e - r_{ed})}} \right]$$

where the plus sign is used if $\dot{r}_{ed} > 0$.

13. Compute the vehicle's velocity vector at departure from

$$V_{sd} = \sqrt{\mu_s \left(\frac{2}{r_{sd}} - \frac{1}{a_s}\right)}$$

14. Compute the elevation angle of V_{sd} from

$$\gamma_{sd} = \cos^{-1}\left[\pm\sqrt{\frac{a_s^2\,(1 - e_s^2)}{r_{sd}\,(2a_s - r_{sd})}}\right]$$

where the plus sign is used if $0 \leq v_{sd} \leq \pi$.

15. Compute i_s, the inclination angle of the transfer orbit plane, from

$$i_s = \tan^{-1}\left[\tan\lambda_{pa}/\sin\left(\Lambda_{pa} - \Lambda_{sd}\right)\right]$$

16. Compute α, the angle between $\overline{V}_{ed}$ and $\overline{V}_{sd}$, from

$$\alpha = \cos^{-1}\left[\sin\gamma_{ed}\sin\gamma_{sd} + \cos\gamma_{ed}\cos\gamma_{sd}\cos i_s\right]$$

17. Compute V_∞ from

$$V_\infty = \sqrt{V_{sd}^2 + V_{ed}^2 - 2V_{sd}V_{ed}\cos\alpha}$$

18. Compute $\Delta V_\infty = V_\infty$ (step 17) - V_∞ (fixed).

If $|\Delta V_\infty| > K$, change t_s (step 2) by an amount

$$\Delta t_s = \Delta V_\infty / X$$

where

$$X = \frac{\partial V_\infty}{\partial V_{sd}}\,\frac{dV_{sd}}{dt_s} + \frac{\partial V_\infty}{\partial\alpha}\,\frac{d\alpha}{dt_s}$$

Note: no attempt is made here to express X in terms of the orbital parameters since the equation is tedious even with several simplifying assumptions. A purely iterative procedure is suggested.

If $|\Delta V_\infty| \leq K$, the orbit is established.

In order for a transfer orbit to be possible, the specified V_∞ must be equal to or greater than the minimum value of V_∞. For transfer to inner or outer planets, $a_s \geq \frac{1}{2}(r_{pp} + r_{ep})$ where the r's are the perihelion distances of the orbits of Earth and the destination planet.

For transfer to outer planets, V_∞ must satisfy the equation

$$V_\infty \geq \sqrt{2\mu\, r_{pp}/r_{ep}\,(r_{pp} + r_{ep})} - \sqrt{\mu\left(\frac{2}{r_{ep}} - \frac{1}{a_e}\right)}$$

For transfer to inner planets

$$V_\infty \geq \sqrt{2\mu\, r_{pa}/r_{ea}\,(r_{ea} + r_{pa})} - \sqrt{\mu\left(\frac{2}{r_{ea}} - \frac{1}{a_e}\right)}$$

where r_{pa} and r_{ea} are the aphelion distances of the planet and Earth.

The geometry of the motion for a special transfer is shown in Fig. 8.

The first ten steps of this method are used to establish an orbit between a known departure point and a known arrival point, which will permit the vehicle to intercept the destination planet. The remaining steps of the methods are used to determine the relative magnitudes of the computed V_∞, which is based on the established transfer orbit, and the specified V_∞.

The choice of the parameter t_d determines the position on the Earth's orbit E_d, S_d. This point remains fixed while v_{sd} and then, if necessary, t_s is varied in order to establish a transfer orbit.

A selection of t_d and t_s determines the positions E_d, S_d, and S_{a1}, which is a position on the destination planet's orbit. Assume that orbit 1 is computed using a_s(min) and that t_s is too large. In this case the planet may be at P_{a1} when the vehicle is at S_{a1}. Orbit 2 is computed using the

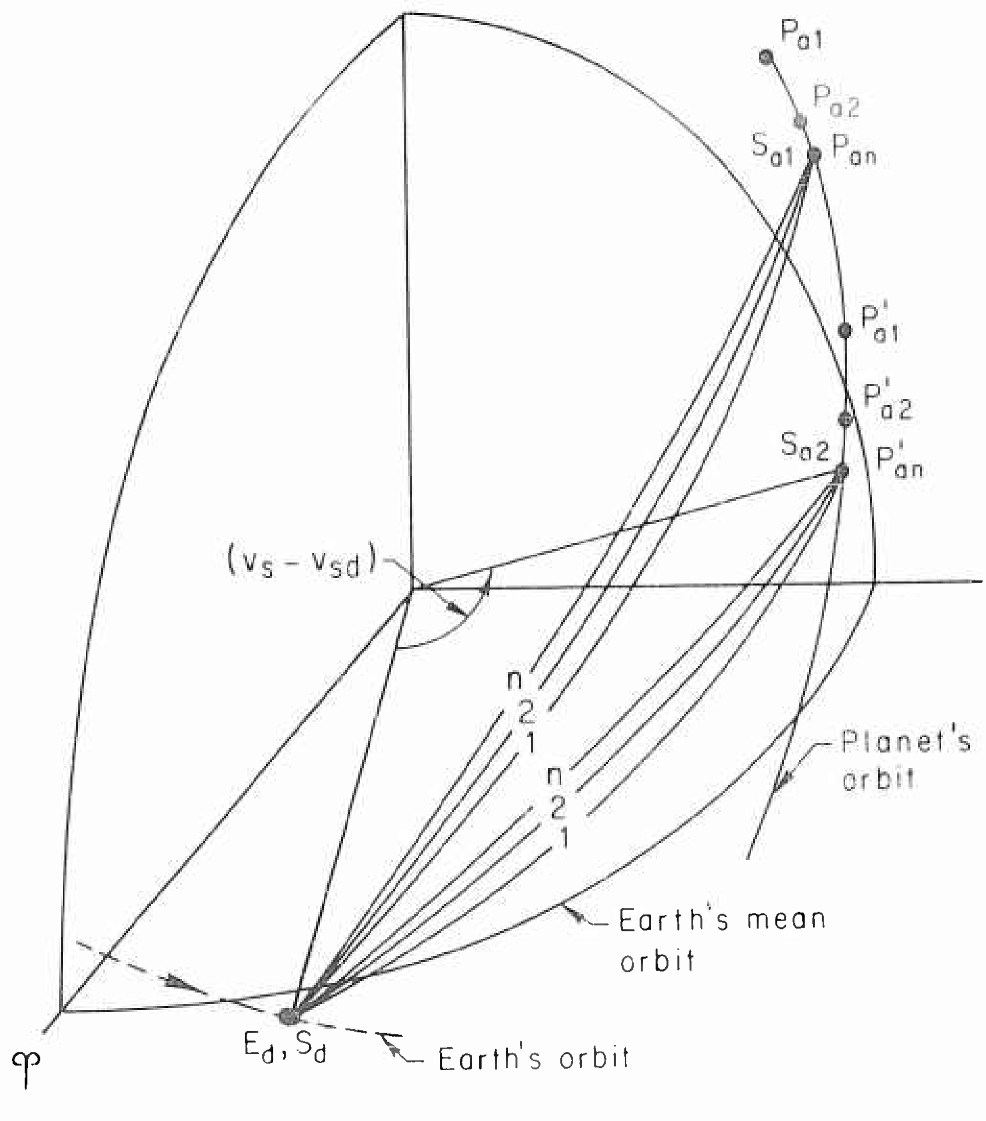

Fig. 8 — Geometry of method D.1

a_s which is based on the coordinates of S_d and S_{a1} and the v_{sd}. The t_s may be such that the planet is at P_{a2} when the vehicle is at S_{a1}. At this point in the computations the v_{sd} is changed and new orbits are computed until the orbit n causes P_{an} and S_{a1} to coincide, i.e., the vehicle intercepts the planet.

For orbit n, which connects S_d and S_{a1}, the required V_∞ is determined and compared to the specified V_∞. If the V_∞'s disagree, then the arrival point on the destination planet orbit is changed by taking a different value of t_s. As before, orbits are computed until the vehicle intercepts the planet at S_{a2}. The V_∞'s are again compared, and if they are different a new value of t_s is determined and the process repeats until an orbit is found which will cause the required V_∞ to match the specified V_∞. If no such orbit can be found then the parameter t_d must be changed.

D.2 FIXED V_∞, PARAMETER t_a

1. Choose t_a and obtain Λ_{pa}, λ_{pa}, and r_{pa}.
2. Choose t_s.
3. Compute $t_d = t_a - t_s$ and obtain Λ_{ed} and r_{ed}.

4-18. These steps are the same as in D.1.

The geometry of motion is different from that of D.1 and is shown in Fig. 9.

In this method the time of arrival is selected, consequently the position of the destination planet at the arrival time. This method is similar to D.1 in that v_{sd} is used as the variable in computing the different orbits which connect the fixed arrival point and a particular departure point on

the Earth's orbit; it differs from D.1 in that a change in t_s causes the departure point to move along the Earth's orbit while the arrival point remains fixed.

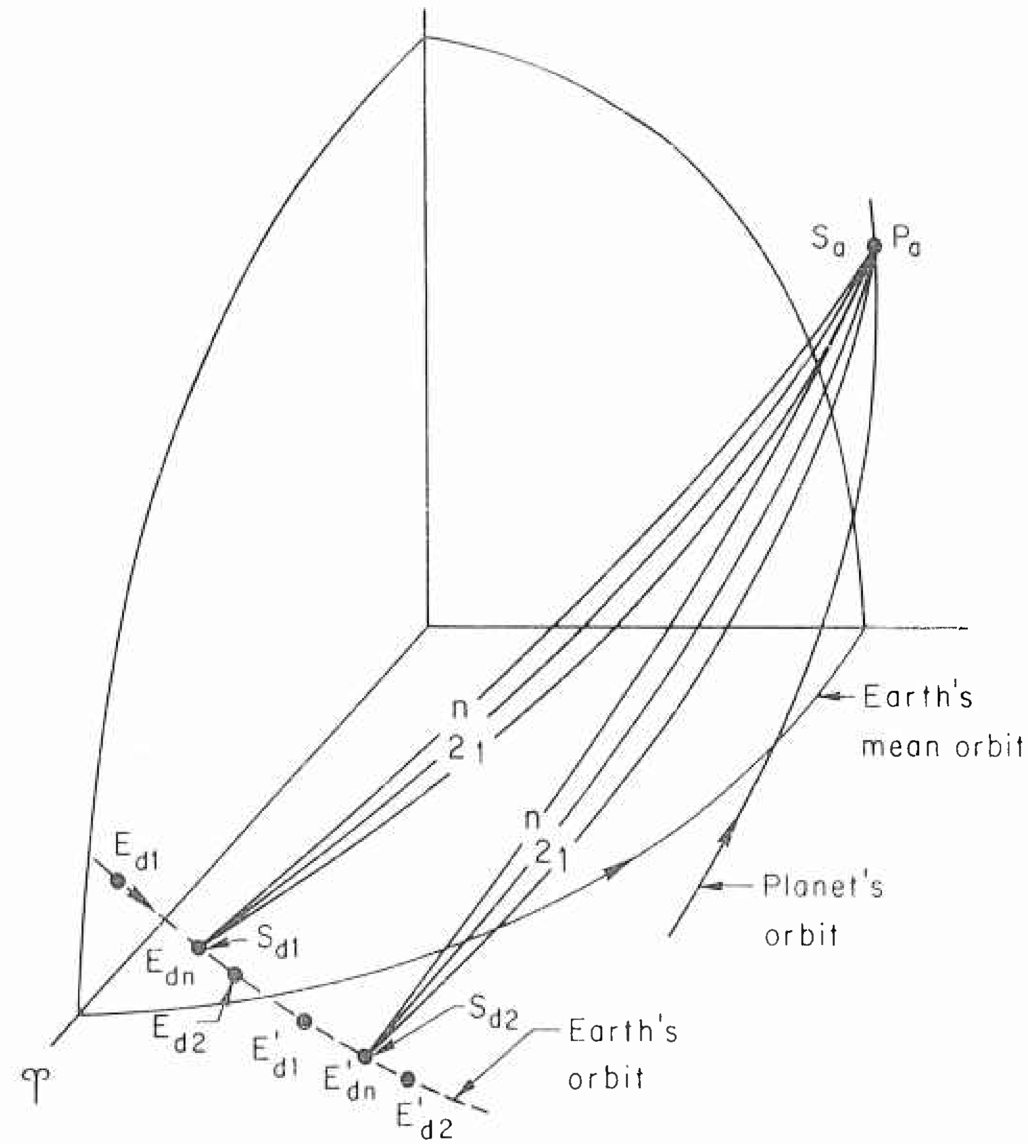

Fig. 9 — Geometry of method D.2

VI. DISCUSSION

The methods presented and discussed here are only a few of many which could be employed to establish interplanetary transfer orbits. These methods are not necessarily better than others, but they represent some of the more obvious approaches to the problem and at the same time involve some of the parameters of greatest interest.

The detailed computation procedure given with each method will be of value in preparing a computer program. The computation procedure plus the discussion and figures will give the reader a "feel" for the relative geometry of motion and spatial orientation of the vehicle and planets as a function of time for each of the various parameters.

Several equations are included in the computation procedures, namely Eqs. (8) - (16), for the purpose of reducing the number of iterations required. These equations are obtained by taking the derivatives of various equations of two-body motion. Since the evaluation of these equations is straightforward, though perhaps tedious, their use may in some cases reduce the computation time that would be required using a purely iterative procedure. The advantage of using these equations rather than a simple trial-and-error approach depends somewhat on the type of computer used.

Appendix

DERIVATION OF EQUATIONS

DERIVATION OF EQ. (8) OF METHOD A.1

According to Eq. (5), t_s is a function of n_s, η, and η_1. Thus

$$\frac{dt_s}{da_s} = \frac{\partial t_s}{\partial n_s}\frac{dn_s}{da_s} + \frac{\partial t_s}{\partial \eta}\frac{d\eta}{da_s} + \frac{\partial t_s}{\partial \eta_1}\frac{d\eta_1}{da_s} \tag{17}$$

where

$$n_s = \frac{k'\sqrt{\mu}}{a_s^{3/2}} = \frac{\sqrt{\mu_s}}{a_s^{3/2}} \tag{18}$$

$$\eta = 2\sin^{-1}\left[\frac{1}{2}\left(\frac{r_{sd} + r_{pa} + c}{a_s}\right)^{1/2}\right] \tag{19}$$

$$\eta_1 = 2\sin^{-1}\left[\frac{1}{2}\left(\frac{r_{sd} + r_{pa} - c}{a_s}\right)^{1/2}\right] \tag{20}$$

and where

$$c = \left[r_{sd}^2 + r_{pa}^2 - 2r_{sd}r_{pa}\cos(v_s - v_{sd})\right]^{1/2} \tag{21}$$

For departures from Earth and heliocentric transfer orbits, $r_{sd} = r_{ed}$ and μ_s is the heliocentric gravitational constant. For convenience the units may be chosen so that $k' = 1$.

The total derivatives are evaluated as follows:

$$\frac{dn_s}{da_s} = \frac{d\left(\dfrac{\sqrt{\mu_s}}{a_s^{3/2}}\right)}{da_s} = -\frac{3\sqrt{\mu_s}}{2a_s^{5/2}} \tag{22}$$

$$\frac{d\eta}{da_s} = \frac{\partial\eta}{\partial r_{sd}}\frac{dr_{sd}}{da_s} + \frac{\partial\eta}{\partial r_{pa}}\frac{dr_{pa}}{da_s} + \frac{\partial\eta}{\partial c}\frac{dc}{da_s} + \frac{\partial\eta}{\partial a_s}$$

Since t_a is the parameter, r_{pa} is constant. We assume that r_{sd} is a constant.

Also, because $(v_s - v_{sd})$ is fixed, c is constant. Thus

$$\frac{d\eta}{da_s} = \frac{\partial\eta}{\partial a_s} = -\frac{1}{a_s}\tan\eta/2 \tag{23}$$

In a similar manner we find that

$$\frac{d\eta_1}{da_s} = \frac{\partial\eta_1}{\partial a_s} = -\frac{1}{a_s}\tan\eta_1/2 \tag{24}$$

The partial derivatives of t_s depend on the type of transfer orbit and are obtained using Eq. (5). The expressions for the partials are

$$\frac{\partial t_s}{\partial n_s} = -\frac{t_s}{n_s} \tag{25}$$

for all types of transfer orbits.

$$\frac{\partial t_s}{\partial\eta} = \pm\frac{1}{n_s}(1 - \cos\eta) \tag{26}$$

where the plus sign is used for direct and perihelion transfer and the minus sign is used for aphelion and indirect transfer.

$$\frac{\partial t_s}{\partial \eta_1} = \pm \frac{1}{n_s} (1 - \cos \eta_1) \qquad (27)$$

where the plus sign is used for perihelion and indirect transfer and the minus sign is used for direct and aphelion transfer.

Substitution of the derivatives into Eq. (17) gives

$$\frac{dt_s}{da_s} = \sqrt{\frac{a_s}{\mu_s}} \left[\frac{3}{2} t_s n_s \pm (1 - \cos \eta) \tan \eta/2 \pm (1 - \cos \eta_1) \tan \eta_1/2\right] \qquad (28)$$

where the multiple signs in the brackets depend on the type of transfer as follows:

Direct = -, + Perihelion = -, -

Aphelion = +, + Indirect = +, -

DERVIATION OF EQ. (9) OF METHOD A.2

$$\frac{d(\Delta v)}{da_s} = \frac{d\left[(v_s - v_{sd}) - (v_s - v_{sd})\ (\text{fixed})\right]}{da_s} = \frac{d(v_s - v_{sd})}{da_s} \qquad (29)$$

$(v_s - v_{sd})$ is related to the coordinates of the departure and destination planets as follows:

$$v_s - v_{sd} = \cos^{-1} \left[\cos \lambda_{pa} \cos (\Lambda_{pa} - \Lambda_{sd})\right] \qquad (30)$$

where $\Lambda_{sd} = \Lambda_{ed}$. Thus

$$\frac{d(v_s - v_{sd})}{da_s} = \frac{\partial(v_s - v_{sd})}{\partial\lambda_{pa}} \frac{d\lambda_{pa}}{da_s} + \frac{\partial(v_s - v_{sd})}{d\Lambda_{pa}} \frac{d\Lambda_{pa}}{da_s} \tag{31}$$

Since t_d is the parameter in this method, Λ_{sd} is constant and

$$\frac{d\Lambda_{sd}}{da_s} = 0$$

Now, the total derivatives of Eq. (31) are

$$\frac{d\lambda_{pa}}{da_s} = \frac{\partial\lambda_{pa}}{\partial t_a} \frac{dt_a}{da_s} = \frac{\partial\lambda_{pa}}{\partial t_a} \frac{d(t_d + t_s)}{da_s} \tag{32}$$

Since t_d is fixed

$$\frac{d\lambda_{pa}}{da_s} = \frac{\partial\lambda_{pa}}{\partial t_a} \frac{dt_s}{da_s}$$

Similarly

$$\frac{d\Lambda_{pa}}{da_s} = \frac{\partial\Lambda_{pa}}{\partial t_a} \frac{dt_s}{da_s}$$

According to Eq. (5), t_s is a function of n_s, η, and η_1, which are given by Eqs. (18) - (20).

A choice of t_d fixes r_{sd}. We assume that r_{pa} is constant; since $(v_s - v_{sd})$ is fixed, c is constant and dt_s/da_s is given by Eq. (28),

which is

$$\frac{dt_s}{da_s} = \sqrt{\frac{a_s}{\mu_s}}\left[\frac{3}{2}\,t_s n_s \pm (1 - \cos\eta)\tan\eta/2 \pm (1 - \cos\eta_1)\tan\eta_1/2\right] \quad (28)$$

The signs of the terms are the same as in the derivation of Eq. (8).

Now, from the geometry

$$\lambda_{pa} = \tan^{-1}\left[\tan i_p \cos(\Lambda_{pa} - \Omega_{pl})\right] \quad (33)$$

and

$$\frac{\partial\lambda_{pa}}{\partial t_a} = \frac{\partial\Lambda_{pa}}{\partial t_a}\tan i_p \cos^2\lambda_{pa}\cos(\Lambda_{pa} - \Omega_p) \quad (34)$$

The partial derivative

$$\frac{\partial\Lambda_{pa}}{\partial t_a} = \dot{\Lambda}_{pa} \simeq \dot{v}_{pa} = \frac{\sqrt{\mu_s P_p}}{r_{pa}^2} \quad (35)$$

and Eq. (34) becomes

$$\frac{\partial\lambda_{pa}}{\partial t_a} = \dot{\lambda}_{pa} \simeq \frac{\sqrt{\mu_s P_p}}{r_{pa}^2}\tan i_p \cos^2\lambda_{pa}\cos(\Lambda_{pa} - \Omega_p) \quad (36).$$

The partial derivatives of $(v_s - v_{sd})$ are found using Eq. (30). They are

$$\frac{\partial(v_s - v_{sd})}{\partial\lambda_{pa}} = \frac{\sin\lambda_{pa}\cos(\Lambda_{pa} - \Lambda_{sd})}{\sin(v_s - v_{sd})} \quad (37)$$

and

$$\frac{\partial(v_s - v_{sd})}{\partial\Lambda_{pa}} = \frac{\cos\lambda_{pa}\sin(\Lambda_{pa} - \Lambda_{sd})}{\sin(v_s - v_{sd})} \tag{38}$$

Substitution of Eqs. (34), (35), (37), and (38) into Eq. (31), after using small-angle approximations, i.e., $\sin\lambda_{pa} = \lambda_{pa}$, $\cos\lambda_{pa} = 1$, and $\tan i_p = i_p$, gives

$$\frac{d(v_s - v_{sd})}{da_s} = \frac{\sqrt{\mu_s P_p}}{r_{pa}^2 \sin(v_s - v_{sd})}\left[\lambda_{pa}\, i_p \cos(\Lambda_{pa} - \Lambda_{sd})\right]\frac{dt_s}{da_s}$$

Since λ_{pa} and i_p are small for the planets of interest, their product is neglected. In accord with this and the assumptions above

$$\sin(v_s - v_{sd}) \simeq \sin(\Lambda_{pa} - \Lambda_{sd})$$

and we get

$$\frac{d(v_s - v_{sd})}{da_s} = \frac{\sqrt{\mu_s P_p}}{r_{pa}^2}\,\frac{dt_s}{da_s}$$

Substituting for dt_s/da_s gives

$$\frac{d(v_s - v_{sd})}{da_s} = \frac{\sqrt{a_s P_p}}{r_{pa}^2}\left[+\frac{3}{2}t_s n_s \pm (1 - \cos\eta)\tan\eta/2 \pm (1 - \cos\eta_1)\tan\eta_1/2\right] \tag{39}$$

where the multiple signs in the brackets depend on the type of transfer path as follows:

Direct = -, +	Perihelion = -, -
Aphelion = +, +	Indirect = +, -

DERIVATION OF Eq. (10) OF METHOD A.3

$$\frac{d(\Delta\Lambda)}{dt_a} = \frac{d(\Lambda_{sd} - \Lambda_{ed})}{dt_a} = \frac{d\Lambda_{sd}}{dt_a} - \frac{d\Lambda_{ed}}{dt_a} \tag{40}$$

Since $(v_s - v_{sd})$ is fixed and v_{sd} is specified, v_s is determined. If, in addition, we assume that r_{sd} and r_{pa} are constant, t_s will also be constant, and from $t_a = t_d + t_s$

$$dt_a = dt_d$$

Eq. (40) becomes

$$\frac{d\Delta\Lambda}{dt_a} = \frac{d\Lambda_{sd}}{dt_a} - \frac{d\Lambda_{ed}}{dt_d} \tag{41}$$

From the geometry we get

$$\Lambda_{sd} = \Lambda_{pa} - \cos^{-1}\left[\cos(v_s - v_{sd})/\cos\lambda_{pa}\right] \tag{42}$$

and

$$\frac{d\Lambda_{sd}}{dt_a} = \dot{\Lambda}_{pa} - \frac{\cos(v_s - v_{sd})\sin\lambda_{pa}}{\sin(\Lambda_{pa} - \Lambda_{sd})\cos^2\lambda_{pa}}\dot{\lambda}_{pa} \tag{43}$$

Again using small-angle approximations, as in the derivation of Eq. (9),

and Eqs. (35) and (36), we substitute into Eq. (43) and get

$$\frac{d\Lambda_{sd}}{dt_a} = \frac{\sqrt{\mu_s P_p}}{r_{pa}^{\;2}} \left[1 - \frac{\lambda_{pa} i_p \cos (v_s - v_{sd}) \cos (\Lambda_{pa} - \Omega_p)}{\sin (\Lambda_{pa} - \Lambda_{sd})} \right] \tag{44}$$

The second term on the right side of Eq. (44) is small compared to one except for values of $(\Lambda_{pa} - \Lambda_{sd})$ near 180°, since the product $\lambda_{pa} i_p$ is small for all planets of interest. For $(\Lambda_{pa} - \Lambda_{sd}) \simeq 180^{\circ}$ the inclination of the transfer orbit plane is approximately 90°. Transfer orbits of this inclination would require extremely high velocities; therefore this term is neglected and

$$\frac{d\Lambda_{sd}}{dt_a} = \frac{\sqrt{\mu_s P_p}}{r_{pa}^{\;2}} \tag{45}$$

The time rate of change of the Earth's heliocentric longitude is given by

$$\frac{d\Lambda_{ed}}{dt_a} = \frac{\sqrt{\mu_s P_e}}{r_{ed}^{\;2}} \tag{46}$$

Substitution of Eqs. (45) and (46) into Eq. (41) gives

$$\frac{d(\Delta\Lambda)}{dt_a} = \sqrt{\mu_s} \left[\frac{\sqrt{P_p}}{r_{pa}^{\;2}} - \frac{\sqrt{P_e}}{r_{ed}^{\;2}} \right] \tag{47}$$

DERIVATION OF EQ. (11) OF METHOD B.3

$$\frac{d(\Delta t_s)}{dt_d} = \frac{d\left[t_s - t_s\ (\text{fixed})\right]}{dt_d} = \frac{dt_s}{dt_d}$$

Since $t_s = f(n_s, \eta, \eta_1)$

$$\frac{dt_s}{dt_d} = \frac{\partial t_s}{\partial n_s} \frac{dn_s}{dt_d} + \frac{\partial t_s}{\partial \eta} \frac{d\eta}{dt_d} + \frac{\partial t_s}{\partial \eta_1} \frac{d\eta_1}{dt_d} \tag{48}$$

The total derivatives are as follows:

$$\frac{dn_s}{dt_d} = -\frac{3\sqrt{\mu_s}}{2a_s^{5/2}} \frac{da_s}{dt_d} \tag{49}$$

If we assume that r_{sd} and r_{pa} are constants and use the fact that v_{sd} is fixed, then using Eq. (1)

$$\frac{da_s}{dt_d} = \frac{\partial a_s}{\partial e_s} \frac{de_s}{dt_d} = \frac{\partial a_s}{\partial e_s} \frac{\partial e_s}{\partial v_s} \frac{dv_s}{dt_d} \tag{50}$$

since $e_s = f(r_{sd}, r_{pa}, v_{sd}, v_s)$. Using Eq. (30)

$$\begin{aligned} \frac{dv_s}{dt_d} &= \frac{\partial v_s}{\partial \lambda_{pa}} \frac{d\lambda_{pa}}{dt_d} + \frac{\partial v_s}{\partial \Lambda_{pa}} \frac{d\Lambda_{pa}}{dt_d} \\ &= \frac{\partial v_s}{\partial \lambda_{pa}} \frac{d\lambda_{pa}}{dt_a} + \frac{\partial v_s}{\partial \Lambda_{pa}} \frac{d\Lambda_{pa}}{dt_a} \end{aligned} \tag{51}$$

since $t_a = t_d + t_s$ and t_s is constant.

The partial derivative $\partial v_s/\partial\lambda_{pa}$ contains the $\sin \lambda_{pa}$, and the total derivative $d\lambda_{pa}/dt_a$ contains $\tan i_p$. Since both λ_{pa} and i_p are small, the product of these derivatives is neglected in Eq. (51) and subsequent steps of the derivation. Thus, Eq. (51) becomes

$$\frac{dv_s}{dt_d} = \frac{\partial v_s}{\partial \Lambda_{pa}} \frac{d\Lambda_{pa}}{dt_a} \tag{52}$$

and Eq. (49) becomes

$$\frac{dn_s}{dt_d} = \frac{-3\sqrt{\mu_s}}{2a_s^{5/2}} \frac{\partial a_s}{\partial e_s} \frac{\partial e_s}{\partial v_s} \frac{\partial v_s}{\partial \Lambda_{pa}} \dot{\Lambda}_{pa} \tag{53}$$

According to the assumptions above and Eq. (19)

$$\frac{d\eta}{dt_d} = \frac{\partial\eta}{\partial a_s} \frac{da_s}{dt_a} + \frac{\partial\eta}{\partial c} \frac{dc}{dt_a}$$

$$= \frac{\partial\eta}{\partial a_s} \frac{da_s}{dt_a} + \frac{\partial\eta}{\partial c} \frac{\partial c}{\partial v_s} \frac{dv_s}{dt_a} \tag{54}$$

Substitution of Eqs. (50) and (52) into Eq. (54) gives

$$\frac{d\eta}{dt_d} = \left(\frac{\partial\eta}{\partial a_s} \frac{\partial a_s}{\partial e_s} \frac{\partial e_s}{\partial v_s} + \frac{\partial\eta}{\partial c} \frac{\partial c}{\partial v_s}\right) \frac{\partial v_s}{\partial \Lambda_{pa}} \dot{\Lambda}_{pa} \tag{55}$$

By using Eq. (20) in a similar manner

$$\frac{d\eta_1}{dt_d} = \left(\frac{\partial\eta_1}{\partial a_s} \frac{\partial a_s}{\partial e_s} \frac{\partial e_s}{\partial v_s} + \frac{\partial\eta_1}{\partial c} \frac{\partial c}{\partial v_s}\right) \frac{\partial v_s}{\partial \Lambda_{pa}} \dot{\Lambda}_{pa} \tag{56}$$

By using Eqs. (19), (20), and (21) we find that

$$\frac{\partial\eta}{\partial a_s} = -\frac{1}{a_s}\tan\eta/2 \qquad \frac{\partial\eta}{\partial c} = \frac{1}{2a_s\sin\eta}$$

$$\frac{\partial\eta_1}{\partial a_s} = -\frac{1}{a_s}\tan\eta_1/2 \qquad \frac{\partial\eta_1}{\partial c} = -\frac{1}{2a_s\sin\eta_1} \tag{56a}$$

$$\frac{\partial c}{\partial v_s} = \frac{1}{c}\, r_{sd} r_{pa}\sin(v_s - v_{sd})$$

By solving Eq. (1) for e_s and then for a_s we get

$$e_s = \frac{r_{sd} - r_{pa}}{r_{pa}\cos v_s - r_{sd}\cos v_{sd}}$$

and

$$a_s = \frac{r_{sd}(1 + e_s\cos v_{sd})}{1 - e_s^2}$$

The partial derivatives are

$$\frac{\partial a_s}{\partial e_s} = \frac{a_s(1 + e_s^2) - r_{sd}}{e_s(1 - e_s^2)}$$

and

$$\frac{\partial e_s}{\partial v_s} = \frac{e_s^2\, r_{pa}\sin v_s}{r_{sd} - r_{pa}}$$

Using Eq. (30) we get

$$\frac{\partial v_s}{\partial \Lambda_{pa}} = \frac{\cos \lambda_{pa} \sin (\Lambda_{pa} - \Lambda_{sd})}{\sin (v_s - v_{sd})}$$

By substituting for the partial derivatives of Eqs. (53), (55), and (56) and then substituting these equations into Eq. (48) and simplifying, we get

$$\frac{dt_s}{dt_d} = \left(\frac{e_s\left[a_s\ (1 + e_s^2) - r_{sd}\right]r_{pa}\ \sin v_s}{(1 - e_s^2)\ (r_{sd} - r_{pa})}\right)\left[\frac{-3\sqrt{\mu_s}}{2a_s^{5/2}}\ \frac{\partial t_s}{\partial n_s}\right.$$

$$\left. - \frac{1}{a_s} \tan \eta/2\ \frac{\partial t_s}{\partial \eta} - \frac{1}{a_s} \tan \eta_1/2\ \frac{\partial t_s}{\partial \eta_1}\right] \dot{\Lambda}_{pa}$$

$$+ \frac{r_{sd} r_{pa}}{c} \sin (v_s - v_{sd}) \left[\frac{1}{2a_s \sin \eta}\ \frac{\partial t_s}{\partial \eta} - \frac{1}{2a_s \sin \eta_1}\ \frac{\partial t_s}{\partial \eta_1}\right] \dot{\Lambda}_{pa}$$

If we eliminate the remaining partial derivatives using Eqs. (25), (26), and (27) and simplify, we get

$$\frac{dt_s}{dt_d} = \frac{\sqrt{a_s P_p}}{r_{pa}} \left\{ \left(\frac{e_s\ a_s\ (1 + e_s^2) - r_{sd}\ \sin v_s}{(1 - e_s^2)\ \ (r_{sd} - r_{pa})}\right)\left[\frac{3}{2} n_s t_s \pm (1 - \cos \eta) \tan \eta/2\right.\right.$$

$$\left.\left. \pm (1 - \cos \eta_1) \tan \eta_1/2\right] + \frac{r_{sd}}{2c} \sin (v_s - v_{sd}) \left[\pm \tan \eta/2 \pm \tan \eta_1/2\right] \right\} \tag{57}$$

The multiple signs in brackets depend on the transfer orbit as follows:

Direct = -, + and +, + Perihelion = -, - and +, -

Aphelion = +, + and -, + Indirect = +, - and -, -

DERIVATION OF Eq. (12) of METHOD C.1

$$\frac{d(\Delta a_s)}{dt_a} = \frac{d\left[a_s(\min) - a_s\right]}{dt_a} = \frac{d\ a_s(\min)}{dt_a} \qquad (58)$$

since for fixed V_{sd} a choice of t_d determines r_{ed} and consequently a_s. Thus

$$\frac{da_s}{dt_a} = 0$$

The equation for $a_s(\min)$ is

$$a_s(\min) = \frac{1}{4}\,(r_{sd} + r_{pa} + c) \qquad (59)$$

where $r_{sd} = r_{ed}$. We assume that r_{pa} is constant. As a result of this

$$\frac{da_s(\min)}{dt_a} = \frac{\partial a_s(\min)}{\partial c}\ \frac{dc}{dt_a}$$

Using Eq. (2) for c and Eq. (30) we get

$$\frac{dc}{dt_a} = \frac{\partial c}{\partial(v_s - v_{sd})} \frac{d(v_s - v_{sd})}{dt_a}$$

$$= \frac{\partial c}{\partial(v_s - v_{sd})} \left[\frac{\partial(v_s - v_{sd})}{\partial\Lambda_{pa}} \dot{\Lambda}_{pa} + \frac{\partial(v_s - v_{sd})}{\partial\lambda_{pa}} \dot{\lambda}_{pa} \right] \quad (60)$$

Since the coefficient of $\dot{\lambda}_{pa}$ contains the sin λ_{pa} and λ_{pa} contains tan i_p, the second term in the bracket is neglected as in the derivation of Eq. (9).

Equation (58) may now be written as

$$\frac{da_s(min)}{dt_a} = \frac{\partial a_s(min)}{\partial c} \frac{\partial c}{\partial(v_s - v_{sd})} \frac{\partial(v_s - v_{sd})}{\partial\Lambda_{pa}} \dot{\Lambda}_{pa} \quad (61)$$

By using Eqs. (21), (30), and (59), the partial derivatives are evaluated and then substituted into Eq. (61). After simplification we have

$$\frac{da_s(min)}{dt_a} = \frac{\sqrt{\mu_s P_p}\; r_{sd}}{4c\; r_{pa}} \sin(\Lambda_{pa} - \Lambda_{sd}) \quad (61a)$$

<u>DERIVATION OF EQ. (13) OF METHOD C.1</u>

The assumptions made in the derivation of Eq. (12) are valid in this derivation

$$\frac{d(\Delta t_s)}{dt_a} = \frac{d\left[t_s(\text{step } 9) - t_s(\text{step } 4)\right]}{dt_a} = \frac{dt_s(\text{step } 9)}{dt_a} - 1 \quad (62)$$

The derivative $dt_s(\text{step } 4)/dt_a = 1$ since for t_d(fixed) a change in the chosen value of t_a causes an equal change in t_s.

$$\frac{dt_s}{dt_a} = \frac{\partial t_s}{\partial n_s}\frac{dn_s}{dt_a} + \frac{\partial t_s}{\partial \eta}\frac{d\eta}{dt_a} + \frac{\partial t_s}{\partial \eta_1}\frac{d\eta_1}{dt_a} \tag{63}$$

For fixed V_{sd} and t_d, a_s is determined and

$$\frac{dn_s}{dt_a} = \frac{d\left(\sqrt{\mu_s}/a_s^{3/2}\right)}{dt_a} = 0$$

Using Eqs. (19), (20), (21), and (60) we get

$$\frac{d\eta}{dt_a} = \frac{\partial \eta}{\partial c}\frac{dc}{dt_a} = \frac{\partial \eta}{\partial c}\frac{\partial c}{\partial(v_s - v_{sd})}\frac{\partial(v_s - v_{sd})}{\partial \Lambda_{pa}}\dot{\Lambda}_{pa} \tag{64}$$

and

$$\frac{d\eta_1}{dt_a} = \frac{\partial \eta_1}{\partial c}\frac{dc}{dt_a} = \frac{\partial \eta_1}{\partial c}\frac{\partial c}{\partial(v_s - v_{sd})}\frac{\partial(v_s - v_{sd})}{\partial \Lambda_{pa}}\dot{\Lambda}_{pa} \tag{65}$$

By replacing the total derivatives and simplifying, Eq. (62) becomes

$$\frac{d(\Delta t_s)}{dt_a} = \frac{\partial c}{\partial(v_s - v_{sd})}\frac{\partial(v_s - v_{sd})}{\partial \Lambda_{pa}}\dot{\Lambda}_{pa}\left[\frac{\partial \eta}{\partial c}\frac{\partial t_s}{\partial \eta} + \frac{\partial \eta_1}{\partial c}\frac{\partial t_s}{\partial \eta_1}\right] - 1$$

$$= \frac{r_{sd}\sqrt{\mu_s P_p}}{c\, r_{pa}}\sin(\Lambda_{pa} - \Lambda_{sd})\left(\frac{\partial \eta}{\partial c}\frac{\partial t_s}{\partial \eta} + \frac{\partial \eta_1}{\partial c}\frac{\partial t_s}{\partial \eta_1}\right) - 1 \tag{66}$$

The partial derivatives of Eq. (66), which are given by Eqs. (26), (27), and (56a), are replaced, and we get

$$\frac{d(\Delta t_s)}{dt_a} = \frac{r_{sd}\sqrt{a_s P_p}}{2c\, r_{pa}} \sin(\Lambda_{pa} - \Lambda_{sd}) \left(\pm \tan \eta/2 \pm \tan \eta_1/2\right) - 1 \quad (67)$$

The signs of the terms in parentheses depend on the type of transfer orbit as follows:

Direct = +, +   Perihelion = +, -

Aphelion = -, +   Indirect = -, -

DERIVATION OF EQ. (16) OF METHOD D.1

By choosing t_s in addition to the parameter t_d, the t_a is determined, and consequently $(v_s - v_{sd})$ is a constant. Again we assume that r_{pa} is constant.

$$\frac{d\Delta t_s}{dv_{sd}} = \frac{dt_s(\text{step } 9)}{dv_{sd}}$$

$$= \frac{\partial t_s}{\partial n_s}\frac{dn_s}{\partial v_{sd}} + \frac{\partial t_s}{\partial \eta}\frac{d\eta}{dv_{sd}} + \frac{\partial t_s}{\partial \eta_1}\frac{d\eta_1}{dv_{sd}} \quad (68)$$

The total derivatives are written as follows:

$$\frac{dn_s}{dv_{sd}} = \frac{d\left(\frac{\sqrt{\mu_s}}{a_s^{3/2}}\right)}{dv_{sd}} = -\frac{3\sqrt{\mu_s}}{2a_s^{5/2}}\frac{da_s}{dv_{sd}}$$

$$\frac{d\eta}{dv_{sd}} = \frac{\partial\eta}{\partial a_s}\frac{da_s}{dv_{sd}}$$

$$\frac{d\eta_1}{dv_{sd}} = \frac{\partial\eta_1}{\partial a_s}\frac{da_s}{dv_{sd}}$$

Substitution into Eq. (68) gives

$$\frac{dt_s}{dv_{sd}} = \left(-\frac{3\sqrt{\mu_s}}{a_s^{5/2}}\frac{\partial t_s}{\partial n_s} + \frac{\partial t_s}{\partial\eta}\frac{\partial\eta}{\partial a_s} + \frac{\partial t_s}{\partial\eta_1}\frac{\partial\eta_1}{\partial a_s}\right)\frac{da_s}{dv_{sd}} \tag{69}$$

$$\frac{da_s}{dv_{sd}} = \frac{\partial a_s}{\partial e_s}\frac{de_s}{dv_{sd}} + \frac{\partial a_s}{\partial v_{sd}}$$

$$= \frac{\partial a_s}{\partial e_s}\left(\frac{\partial e_s}{\partial v_s}\frac{dv_s}{dv_{sd}} + \frac{\partial e_s}{\partial v_{sd}}\right) + \frac{\partial a_s}{\partial v_{sd}}$$

Since $(v_s - v_{sd})$ is constant, $dv_s = dv_{sd}$ and

$$\frac{da_s}{dv_{sd}} = \frac{\partial a_s}{\partial e_s}\left(\frac{\partial e_s}{\partial v_s} + \frac{\partial e_s}{\partial v_{sd}}\right) + \frac{\partial a_s}{\partial v_{sd}} \tag{70}$$

where

$$\frac{\partial a_s}{\partial e_s} = \frac{a(1 + e_s^2) - r_{sd}}{e_s(1 - e_s^2)}$$

$$\frac{\partial e_s}{\partial v_s} = \frac{e_s^2\, r_{pa} \sin v_s}{r_{sd} - r_{pa}}$$

$$\frac{\partial e_s}{\partial v_{sd}} = - \frac{e_s^2\, r_{sd} \sin v_{sd}}{r_{sd} - r_{pa}}$$

and

$$\frac{\partial a_s}{\partial v_{sd}} = - \frac{e_s r_{sd} \sin v_{sd}}{1 - e_s^2}$$

Eq. (70) becomes

$$\frac{da_s}{dv_{sd}} = \frac{e_s}{1 - e_s^2} \left[\frac{a(1 + e_s^2) - r_{sd}}{r_{sd} - r_{pa}} \; (r_{pa} \sin v_s - r_{sd} \sin v_{sd}) - r_{sd} \sin v_{sd} \right] \tag{71}$$

The partial derviatives of Eq. (69) are given by Eqs. (23) - (27). Using these equations and Eq. (71), Eq. (69) can be written as

$$\frac{dt_s}{dv_{sd}} = \sqrt{\frac{a_s}{\mu_s}} \; \frac{e_s}{1 - e_s^2} \left[\frac{a_s\, (1 + e_s^2) - r_{sd}}{r_{sd} - r_{pa}} \; (r_{pa} \sin v_s - v_{sd} \sin v_{sd}) - r_{sd} \sin v_{sd} \right] \tag{72}$$

$$\times \left[+ \frac{3}{2}\, n_s t_s \pm (1 - \cos \eta) \tan \eta/2 \pm (1 - \cos \eta_1) \tan \eta_1/2 \right]$$

The multiple signs in the second brackets depend on the type of transfer orbit as follows:

Direct = -, +	Perihelion = -, -
Aphelion = +, +	Indirect = +, -

REFERENCES

1. Schechter, H. B., Determination of Interplanetary Transfer Orbits for Specified Date of Departure, The RAND Corporation, Paper P-2157, December 1, 1960.

2. Breakwell, J. V., R. W. Gillespie, and S. Ross, Researches in Interplanetary Transfer, ARS Preprint 954-59, 14th Annual Meeting, November 16, 1959.

3. Battin, R. H., "The Determination of Round-Trip Planetary Reconnaissance Trajectories," J. Aero/Space Sci, Vol. 26, No. 9, September 1959.